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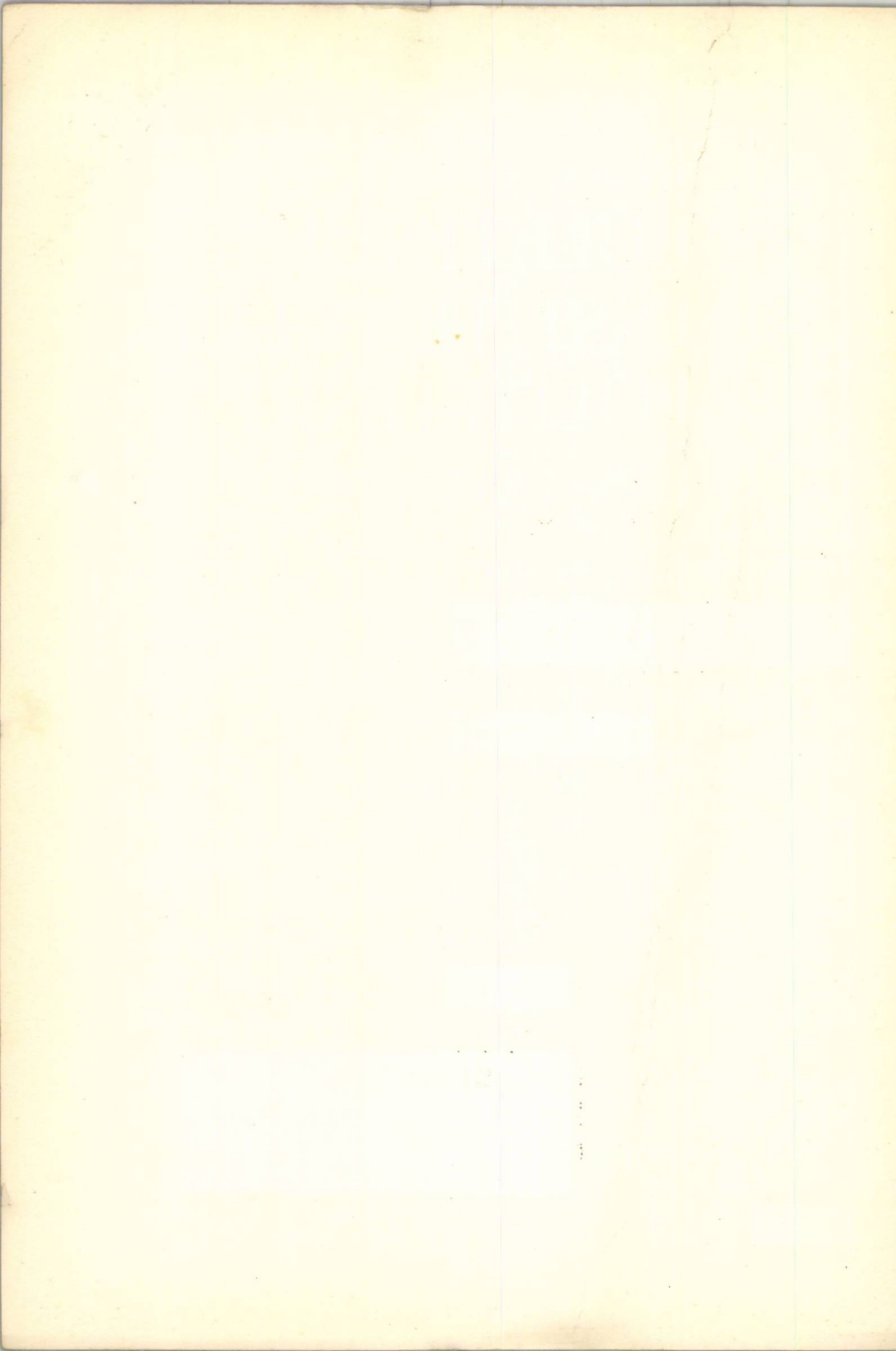
Tomul XXVI (XXX)

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Secția I

MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ

1980





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Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1980

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Tomul XXVI (XXX), Fasc. 1—2

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MATEMATICĂ.
MECANICĂ TEORETICĂ. FIZICĂ

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Secția I

MATEMATICĂ

MECANICĂ TEORETICĂ. FIZICĂ

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PROBLÈMES CONCERNANT CERTAINES EXTENSIONS DES ÉQUATIONS
POLYVIBRANTES, DITES ÉQUATIONS DE MANGERON

PAR

B. A. BONDARENKO, D. MANGERON, J.P. DE OLIVEIRA SANTOS et E. SHIMEMURA

À la Mémoire de l'Académicien

Alexandre T. MYLLER

pour le centenaire de sa naissance

1. Il y a à peu près un demi-siècle lorsque l'un des auteurs a exposé dans le cadre du Séminaire d'Analyse Supérieure dirigé à l'Université de Naples par Mauro Picone, Maître à nous tous, le problème des spectres concernant le système fonctionnel qui généralise celui de Sturm-Liouville :

$$(1) \quad \begin{cases} [A(x)u'(x) + pB(x)u(x)]' + p[B(x)u'(x) + C(x)u(x)] = 0, \\ x = (x_1, x_2, \dots, x_m), \quad u(x) \text{FrR} = 0, \\ u(x), A(x), B(x), C(x) = u(x_1, x_2, \dots, x_m), A(x_1, x_2, \dots, x_m), \dots \end{cases}$$

dont la nouveauté consiste dans le fait que l'opérateur $' \equiv M_m \equiv \partial^m / \partial x_1 \partial x_2 \dots \partial x_m$ et le domaine R est un hyperrectangle à m dimensions $R^1 = \{a_i \leq x_i \leq b_i, i = 1, 2, \dots, m\}$, tandis que $\text{FrR} \equiv \partial R$ est sa frontière. D'où la naissance des équations que son auteur a appelé *équations polyvibrantes*, tandis que tant d'autres les ont nommées *équations de Mangeron*, qui ont des nombreuses applications, telles, par ex., concernant l'„automatic design of free form surfaces“ (Garrett Birkhoff et William Gordon), „interpolation by triangles and tetrahedrons“ (George M. Nielsen, L. Schumacher), „mathematical models of certain phenomena occurring in cold plasma“ (M.N. Oguztoreli).

Depuis lors nombre de mémoires et de notes ont été publiés par l'auteur des équations polyvibrantes, seul ou bien en collaboration avec des représentants de l'École kirgize d'équations intégrodifférentielles, le prof. L.E. Krivochéine en tête, de l'Ecole ouzbèke d'approximation par de nouvelles classes de polynômes, le prof. B.A. Bondarenko en tête, de l'une des écoles canadiennes de mathématiques appliquées et de programmation aux ordinateurs, le prof. M.N. Oguztoreli et le feu prof. K.V. Leung en tête et d'autres encore.

2. Dans quelques-uns de travaux des deux premiers auteurs on a considéré différents problèmes „bien posés“ au sens d'Hadamard¹⁾ pour les équations aux dérivées partielles

$$(2) \quad (M_m^r + L_y)^n u(x, y) = f(x, y),$$

où l'opérateur M_m est l'opérateur polyvibrant ci-dessus, tandis que

¹⁾ Voir, par contre, les travaux du Colloque international, dirigé par M. le Prof. M.Z. Nashed, tenu le 2 — 6 octobre 1979 à l'University of Delaware concernant les *Ill-Posed Problems: Theory and Practice*.

l'opérateur différentiel L_y porte sur un autre ensemble de variables $y = (y_1, y_2, \dots, y_n)$. Pour cette nouvelle classe d'équations on a construit différents systèmes de solutions exactes et en particulier de certains polynômes spéciaux.

Considérons maintenant dans ce qui suit la classe d'équations polyvibrantes de Mangéron où l'on ne suppose plus des ensembles séparés de variables $x = (x_1, x_2, \dots, x_m)$ et $y = (y_1, y_2, \dots, y_n)$. Telle, par exemple, la classe d'équations d'ordre supérieur aux opérateurs suivants

$$\mathfrak{M}_r \equiv a_r M_m^r + a_{r-1} M_m^{r-1} + \dots + a_1 M_m + a_0,$$

où a_k ($k = 1, 2, \dots, r$) sont des nombres constants.

Bornons nous, afin de simplifier l'écriture au cas de deux variables indépendantes x, y . Pour $a_r = 1$, ce que l'on peut toujours faire sans diminuer la généralité, considérons l'équation polyvibrante

$$(3) \quad \mathfrak{M}_r u(x, y) \equiv (M_2^r + \mathfrak{M}_{r-1})u(x, y) = f(x, y), \quad M_2 \equiv \partial^2 / \partial x \partial y$$

et ses itérées, à savoir

$$(4) \quad \mathfrak{M}_r^N u(x, y) \equiv (M_2^r + \mathfrak{M}_{r-1})^N u(x, y) = f(x, y),$$

où N est un nombre entier ≥ 1 et $f(x, y)$ est une fonction connue suffisamment lisse de variables réelles x et y . On peut formuler pour les équations (3) et (4) différents problèmes à la frontière „bien posés“, y inclus le problème de Goursat et d'autres encore.

Avant d'examiner le problème de la résolution concernant les équations (3) et (4), il est nécessaire de construire et d'étudier, sans en tenir compte des conditions auxquelles doivent satisfaire les solutions cherchées, les classes de solutions exactes des équations (3) et (4), homogènes ($f(x, y) = 0$) et non homogènes.

Considérons dans ce but la série „formelle“

$$(5) \quad H_{r,s}^{n,q}(x, y) = \sum_{i=q}^{\infty} (-1)^{i-q} \binom{i}{q} \mathfrak{M}_{r-1}^{i-q} \left[\frac{x^{ri+s}}{(ri+s)!} \frac{y^{n+ri-s}}{(n+ri-s)!} \right], \quad q = 0, 1, \dots, N-1.$$

En appliquant à l'opérateur $\mathfrak{M}_{r-1}^{i-q} = (a_{r-1} M_2^{r-1} + \dots + a_0)^{i-q}$ le théorème polynominle de décomposition, [1], [2], on démontre, après avoir exécuté les transformations correspondantes, tout en tenant compte des inégalités connues, que la série (5) converge uniformément et absolument, non seulement pour $|x| \leq 1$, $|y| \leq 1$, mais aussi pour $|x| > 1$, $|y| > 1$. Par conséquent, la série (5) est dérivable et intégrable terme à terme par rapport à x et à y . Par suite, tant $H_{r,s}^{n,q}(x, y)$ que $H_{r,s}^{n,q}(y, x)$ représentent une nouvelle classe de fonctions entières transcendentes, associées aux équations (3) et (4). Il est à remarquer que dans (5) les nombres $n \geq 0$, $q \geq 0$, $r \geq 1$ et $s \geq 0$ sont des entiers. Les fonctions (5) possèdent nombre de propriétés analytiques et computérielles assez remarquables et commodes.

Écrivons ci dessous quelques-unes des égalités et des relations concernant les fonctions (5). Commençons avant tout par les formules de différentiation, sans y donner ici la démonstration correspondante.

$$(6) \quad \frac{\partial}{\partial x} H_{r,s}^{n,q}(x,y) = H_{r,s-1}^{n,q}(x,y), \quad s \neq 0,$$

$$(7) \quad \frac{\partial}{\partial x} H_{r,0}^{n,q}(x,y) = -\mathfrak{M}_{r-1} H_{r,r-1}^{n+r,q}(x,y) + H_{r,r-1}^{n+r,q-1}(x,y),$$

$$(8) \quad \frac{\partial}{\partial y} H_{r,s}^{n,q}(x,y) = H_{r,s}^{n-1,q}(x,y), \quad n \neq 0, \quad s \leq n + rq - 1.$$

En vertu de ces formules et de la commutativité des opérateurs $\partial/\partial x$, $\partial/\partial y$, $\mathfrak{M}_{r-1}$, on trouve les égalités

$$(9) \quad M_2^r H_{r,s}^{n,q}(x,y) = -\mathfrak{M}_{r-1} H_{r,s}^{n,q}(x,y) + H_{r,s}^{n,q-1}(x,y),$$

$$(10) \quad \mathfrak{M}_{r-1} H_{r,s}^{n,q}(x,y) = \mathfrak{M}_{r-1} H_{r,s}^{n,q}(x,y).$$

En additionnant membres aux membres les égalités (9) et (10), on obtient la relation

$$(11) \quad \mathfrak{M}_r H_{r,s}^{n,q}(x,y) = (M_2^r + \mathfrak{M}_{r-1}) H_{r,s}^{n,q}(x,y) = H_{r,s}^{n,q-1}(x,y),$$

valable pour tous n , $q \geq 1$, $r \geq 1$ ($s = 0, 1, \dots$). En particulier, pour $q = 0$ et $s = 0, 1, \dots, r - 1$, on a $\mathfrak{M}_r H_{r,s}^{n,0}(x,y) = 0$.

Si l'on en tient compte de la relation

$$(12) \quad \mathfrak{M}_r^{q+1} H_{r,r+m}^{n+r,q}(x,y) = \mathfrak{M}_r H_{r,r+m}^{n+r,0}(x,y) = \frac{x^m}{m!} \frac{y^n}{n!},$$

on en déduit des égalités (9) – (12) les théorèmes suivants :

1°. Les fonctions $H_{r,s}^{n,q}(x,y)$ satisfont, pour $q = 0, 1, \dots, N - 1$ et $s = 0, 1, \dots, r - 1$, dans R_2 , l'équation (4) homogène ($f(x,y) = 0$).

2°. Les fonctions $H_{r,r+m}^{n+r,N-1}(x,y)$ sont des solutions particulières de l'équation (4) pourvu que $f(x,y) = (x^m/m!)(y^n/n!)$.

3°. La série $\sum_{m,n=0}^{\infty} a_{m,n} H_{r,r+m}^{n+r,N-1}(x,y)$ est, dans R_2 , une solution de l'équation (4) et cette série y converge uniformément et absolument chaque fois que converge absolument et uniformément la série $f(x,y) = \sum_{m,n=0}^{\infty} a_{m,n} \frac{x^m}{m!} \frac{y^n}{n!}$.

3. Remarques. a. Pour $N=1$ les théorèmes 1°–3° sont valables pour les équations de Mangéron (3).

b. La démonstration des théorèmes 1° – 3° est fondée sur les propriétés (6) – (12).

c. Les équations (3) et (4) possèdent, pour $a = 0$, des solutions polynomiales.

d. Les fonctions (5) peuvent être utilisées pour la résolution de différents problèmes se référant aux équations (3) et (4).

e. Les résultats acquis peuvent être étendus aux équations poly-vibrantes à un nombre quelconque de variables indépendantes.

f. On en trouve nombre de résultats nouveaux concernant les extensions dans des espaces abstraites des équations de Mangéron dans un de travaux prochains dû à M. Dicesar Lass Fernandez²⁾.

²⁾ Voir ce Bulletin, pp. 49–54.

Dans l'une de notre prochaine Note on exposera nombre de résultats concernant d'autres extensions des équations polyvibrantes.

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PROBLEME REFERITOARE LA UNELE EXTINDERI ALE ECUAȚIILOR POLIVIBRANTE, NUMITE „ECUAȚII MANGERON“

(Rezumat)

Autorii stabilesc un număr de teoreme referitoare la ecuațiile (3) și (4), care constituie noi extensiuni ale ecuațiilor introduse de D. Mangeron acum aproape o jumătate de secol și cristalizate inițial în problema spectrelor pentru sistemul (1).

ASUPRA UNEI CLASE DE ALGEBRE ASOCIATIVE DE ORDIN CINCI CU SCALARI REALI

DE

I. ENESCU

1. Ne propunem determinarea, pînă la izomorfism, a tipurilor de algebre de ordin cinci cu scalari reali, asociative, comutative și cu unitate, care conțin ca subalgebră proprie algebra lui Sobrero dată, de exemplu, prin tabela [1]

(1)

	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	0	0
e_3	0	0	0	0
e_4	0	0	0	0

(0)

2. În [2], [3], [4], au fost determinate toate algebrele reale de ordin cinci asociative, comutative și cu unitate, găsindu-se în total 26 de tipuri neizomorfe. Desigur că, printre acestea trebuie să căutăm algebrele care constituie obiectul acestei note.

Începem prin a observa că algebra (1) are un singur idempotent diferit de zero care este unitate și, în consecință, este indecompozabilă avînd structura de sumă (nedirectă) dintre corpul complex și algebra nulă de ordinul doi (radicalul) [1]. Atunci, o condiție necesară ca o algebră de ordinul cinci cu proprietățile indicate să conțină, ca subalgebră de ordin patru, algebra lui Sobrero, este ca ea să conțină corpul complex și respectiv algebra nulă de ordin doi. Dar, această condiție nu este suficientă, așa cum se va vedea mai departe.

Pe de altă parte, în [4] și [5] s-a arătat că nu există algebre cu scalari reali, asociative, comutative și cu unitate, indecompozabile, de ordin impar, care să conțină corpul complex ca subalgebră. Acest fapt arată că, pentru rezolvarea problemei puse, este suficient să cercetăm numai algebrele reale de ordin cinci, asociative, comutative și cu unitate, decompozabile, determinate în [3].

Ținând seama de condiția necesară semnalată mai înainte, rămân de cercetat numai cinci tipuri de algebre date respectiv, prin tabelele [3]:

		e_1	e_2	e_3	e_4	e_5			e_1	e_2	e_3	e_4	e_5
	e_1	e_1	e_2	0	0	0		e_1	e_1	e_2	0	0	0
(2)	e_2	e_2	$-e_1$	0	0	0	(3)	e_2	e_2	$-e_1$	0	0	0
	e_3	0	0	e_3	0	0		e_3	0	0	e_3	e_4	0
	e_4	0	0	0	e_4	0		e_4	0	0	e_4	$-e_3$	0
	e_5	0	0	0	0	e_5		e_5	0	0	0	0	e_5
		e_1	e_2	e_3	e_4	e_5			e_1	e_2	e_3	e_4	e_5
	e_1	e_1	e_2	0	0	0		e_1	e_1	0	0	0	0
(4)	e_2	e_2	$-e_1$	0	0	0	(5)	e_2	0	e_2	e_3	e_4	e_5
	e_3	0	0	e_3	e_5	e_5		e_3	0	$-e_2$	e_5	$-e_4$	
	e_4	0	0	e_4	0	0		e_4	0	e_4	e_5	0	0
	e_5	0	0	e_5	0	0		e_5	0	e_5	$-e_4$	0	0
		e_1	e_2	e_3	e_4	e_5			e_1	e_2	e_3	e_4	e_5
	e_1	e_1	e_2	0	0	0		e_1	e_1	0	0	0	0
	e_2	e_2	0	0	0	0		e_2	0	e_2	0	0	0
	e_3	0	0	e_3	e_4	0		e_3	0	0	e_3	e_4	0
	e_4	0	0	e_4	$-e_3$	0		e_4	0	0	$-e_3$	e_4	0
	e_5	0	0	0	0	e_5		e_5	0	0	0	0	e_5

Algebra (2) este polinomială generată de $\varphi(\lambda) = (\lambda^2 + 1)(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, λ_j reali, $j = \overline{1,3}$, fiind deci semisimplă [3]. Radicalul său este format numai cu elementul nul și, în consecință, ea nu poate conține algebra (1). Algebra (3) este, de asemenea, semisimplă fiind polinomială generată de $\varphi(\lambda) = (\lambda - \lambda_1)(\lambda^2 + a^2)(\lambda^2 + b^2)$, λ_1, a, b , reali [3] și, după raționamentul precedent, ea nu poate conține algebra lui Sobrero. Algebra (6) este polinomială generată de $\varphi(\lambda) = (\lambda^2 + 1)(\lambda - \lambda_1)^2(\lambda - \lambda_2)$, λ_1, λ_2 reali [3], fiind suma directă dintre respectiv corpul complex, algebra numerelor duale și corpul real. Radicalul ei, ca sumă directă a radicalilor fiecărei componente a sumei directe [1], este format numai cu radicalul algebrei numerelor duale care este algebra nulă de ordinul unu. Acest fapt arată că algebra (6) nu conține algebra nulă de ordinul doi și, prin urmare, ea nu poate conține algebra lui Sobrero.

Fie acum algebra (4). Prin calcul direct, se constată că ea are unitate, anume elementul $e = e_1 + e_3$, elementele e_1 și e_3 fiind idempotenți ortogonali. Pe de altă parte, un calcul simplu dovedește că în algebra (4) nu există nici un element x cu proprietatea $x^2 = -e$, fapt care arată că unitatea corpului complex nu coincide cu unitatea algebrei (4) în care este conținut.

De altfel, aceasta se vede și direct din tabela (4) unde e_1 (unitatea corpului complex) este diferit de elementul $e = e_1 + e_3$.

Schimbînd baza în (4) după: $u_1 = e_1 + e_3$, $u_2 = e_1$, $u_3 = e_2$, $u_4 = e_4$, $u_5 = e_5$, se obține tabela

	u_1	u_2	u_3	u_4	u_5
u_1	u_1	u_2	u_3	u_4	u_5
u_2	u_2	u_2	u_3	0	0
u_3	u_3	u_3	$-u_2$	0	0
u_4	u_4	0	0	0	0
u_5	u_5	0	0	0	0

care pune în evidență faptul că în algebra (4) există un ideal propriu, fie el I , evident maximal, în care o bază este formată din elementele u_j , $j = 2, 5$. Idealul I este suma directă dintre corpul complex și algebra nulă de ordinul doi, așa cum se vede direct din tabela (4').

Se constată prin calcul direct că idealul I nu are unitate. Aceste două observații arată că idealul I și algebra lui Sobrero nu sînt izomorfe.

Observăm că în idealul I este conținut elementul u_2 , adică unitatea corpului complex care, în același timp, este unitatea algebrei lui Sobrero.

Dacă algebra lui Sobrero ar fi conținută în algebra (4'), atunci, pentru orice element x din algebra (1) ar avea loc relația

$$(7) \quad u_2 x = x \in I.$$

Dar, u_2 este unitatea algebrei lui Sobrero și, atunci, relația (7) implică faptul că algebra (1) este conținută în idealul I . Cum algebrele (1) și I au același ordin, ar urma că ele coincid, ceea ce este imposibil. Am arătat astfel că algebra (4) nu conține algebra lui Sobrero.

În sfîrșit, să cercetăm algebra (5). Ea este polinomială, generată de $\varphi(\lambda) = \lambda(\lambda^2 + 1)^2$, fiind suma directă dintre corpul real și algebra lui Sobrero [3].

Rezultatele stabilite demonstrează următoarea

Te o r e m ă. *Există un singur tip neizomorf de algebră reală de ordin cinci, asociativă, comutativă și cu unitate care conține algebra lui Sobrero și aceasta este algebra polinomială generată de $\varphi(\lambda) = \lambda(\lambda^2 + 1)^2$, putînd fi reprezentată prin tabela (5).*

3. În încheiere observăm următoarele:

1) Orice algebră de ordin cinci cu scalari reali, asociativă, comutativă și cu unitate, este unic determinată dacă ea conține algebra lui Sobrero.

2) În orice algebră reală de ordin cinci, asociativă, comutativă și cu unitate care conține algebra lui Sobrero, aceasta din urmă este ideal maximal.

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SUR UNE CLASSE D'ALGÈBRES ASSOCIATIVES, D'ORDRE CINQ,
À SCALAIRES RÉELS

(Résumé)

On recherche, à l'isomorphisme près, les algèbres réelles d'ordre cinq, associatives, commutatives et à unité, qui contiennent l'algèbre de Sobrero, donnée par la table (1). On trouve qu'il existe un seul type de telles algèbres, à savoir l'algèbre polynomiale générée par $\varphi(\lambda) = \lambda(\lambda^2 + 1)^2$, donnée par la table (5).

Dar, w_3 este unitatea algebrei lui Sobrero și, atunci, relația (7) implică faptul că algebra (1) este conținută în idealul λ . Cum algebra (1) și λ au același ordin, ar urma că ele coincid, ceea ce este imposibil. Am arătat astfel că algebra (4) nu conține algebra lui Sobrero.

În sfârșit, să cercetăm algebra (5). Ea este polinomială, generată de $\varphi(\lambda) = \lambda(\lambda^2 + 1)^2$, fiind suma directă dintre corpul real și algebra lui Sobrero [3].

Rezultatele stabilite demonstrează următoarele:

Teoremă. Există un singur tip nontrivial de algebre reale de ordin cinci, asociative, commutative și cu unitate care conține algebra lui Sobrero și aceasta este algebra polinomială generată de $\varphi(\lambda) = \lambda(\lambda^2 + 1)^2$, putând fi reprezentată prin tabela (5).

3. În încheiere observăm următoarele:

(1) Orice algebra de ordin cinci cu scalari reali, asociativă, commutativă și cu unitate, este unic determinată dacă ea conține algebra lui Sobrero.

(2) În orice algebra reală de ordin cinci, asociativă, commutativă și cu unitate care conține algebra lui Sobrero, aceasta din urmă este ideal maximal.

THE SIMULATION RELATIONS BETWEEN THE CLASSES OF ALGORITHMS

BY

MARIA CAZACU

In the present paper, starting from the general notion of algorithm introduced in [2] a simulation relation is underlined between algorithms and classes of algorithms and on prove that the classes of algorithms $\mathcal{Q}_n^k$, $\mathcal{T}_n^k$, $\mathcal{M}_n$, $\mathcal{O}$ and $\mathcal{R}$ are similar. The concepts and notations introduced in [2] are assumed to be known.

Let I and J be two statements on the sets of states S_1 and S_2 respectively and let $\alpha: S_1 \rightarrow S_2$ be a code. The statement I is α -simulated by statement J (notation $I \leq J(\alpha)$) if $I^* \leq J^*(\alpha, \alpha)$ and $I^\delta \leq J^\delta(\alpha, 1_D)$ (see fig. 1).

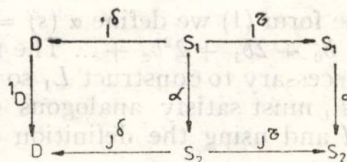


Fig. 1

If the code α is injective and I is total then the relation $I \leq J(\alpha)$ means $J^*(\alpha(s)) = \alpha(I^*(s))$ and $J^\delta(\alpha(s)) = I^\delta(s)$ for any $s \in S_1$.

The statement I is simulated by statement J (notation $I \leq J$) if there is a code α so that $I \leq J(\alpha)$. The simulation relation between the statements is reflective and transitive.

The α -simulation and the simulation relations are transferred to the algorithms by the statements defined by it. Thus, if A and B are algorithms, the relations $A \leq B(\alpha)$ and $A \leq B$ mean $\bar{A} \leq \bar{B}(\alpha)$ and $\bar{A} \leq \bar{B}$ respectively.

Let $\mathcal{I}$ and $\mathcal{J}$ be two sets of statements on the sets of states S_1 and S_2 respectively and let $\alpha: S_1 \rightarrow S_2$ be a code. The set $\mathcal{I}$ is α -simulated by the set $\mathcal{J}$ (notation $\mathcal{I} \leq \mathcal{J}(\alpha)$) if for any $I \in \mathcal{I}$ there is $J \in \mathcal{J}$ so that $I \leq J(\alpha)$. The set $\mathcal{I}$ is simulated by $\mathcal{J}$ (notation $\mathcal{I} \leq \mathcal{J}$) if exists a code α so that $\mathcal{I} \leq \mathcal{J}(\alpha)$. The simulation relation between the sets of statements is also reflective and transitive.

It can be shown

Theorem 1. If $A = (X, U, x_1, \omega, \beta)$, $B = (X, U, x_1, \omega, \beta')$ and $\beta_x \leq \beta'_x$ for any $x \in X$, then $\bar{A} \leq \bar{B}(\alpha)$.

In consequence of this theorem, to obtain an algorithm which α -simulated a given algorithm, it is enough that in the nodes of this, to arrange new components which α -simulates the above mentioned.

From this theorem results that if $\mathcal{I} \leq \mathcal{J}(\alpha)$ then $\bar{\mathcal{I}} \leq \bar{\mathcal{J}}(\alpha)$.

We name two sets of statements $\mathcal{I}$ and $\mathcal{J}$ alike (notation: $\mathcal{I} \approx \mathcal{J}$ if $\mathcal{I} \leq \mathcal{J}$ and $\mathcal{J} \leq \mathcal{I}$).

It can be shown

Theorem 2. If the sets of statements $\mathcal{I}$ and $\mathcal{J}$ are closed and the set $\mathcal{J}$ α -simulates a complete system in $\mathcal{I}$, then $\mathcal{J}$ α -simulates $\mathcal{I}$.

This theorem allows the underlining of some simulation (similarity) relations between different classes of algorithms. We shall consider the above mentioned classes of algorithms.

Theorem 3. $\mathcal{T}_2 \leq \mathcal{O}$.

Proof. Let $\{0,1\}$ be the alphabet of Turing algorithms of $\mathcal{T}_2$ and let S be the set of states for these algorithms. A state $s \in S$ will be by the form

$$(1) \quad s = \dots a_2 a_1 a_0 \times b_0 b_1 b_2,$$

where $a_i, b_i \in \{0,1\}$ and all but finitely many a_i, b_i are 0. Let $S_1 = \{2^a 3^b 5^c \mid a, b, c \in N\}$ be the set of states for the operatorial algorithms. We know that $\mathcal{T}_2 = \text{ALG}(\{L, R, T\})$. In consequence of the Theorem 2 it will be sufficient to show that exists a code $\alpha: S \rightarrow S_1$ and the operatorial statements L, R, T which α -simulates L, R, T .

For any $s \in S$ of the form (1) we define $\alpha(s) = 2^a 3^b 5^c$ where $a = a_0 + 2a_1 + 2^2 a_2 + \dots$, $b = b_0 + 2b_1 + 2^2 b_2 + \dots$. The function α is a injective code. Hence forth it is necessary to construct L_1 so that $L_1(\alpha(s)) = \alpha(L^\tau(s))$ and $L_1^\delta(\alpha(s)) = L^\delta(s)$. R_1, T_1 must satisfy analogous equalities. By the work of the statements L, R, T and using the definition of α , it follows L_1, R_1, T_1 must satisfy the relations

$$L_1(2^a 3^b) = \begin{cases} (2^{\frac{a}{2}} 3^{2b}, 0) & \text{for even } a, \\ (2^{\frac{a-1}{2}} 3^{2b+1}, 1) & \text{for odd } a; \end{cases}$$

$$R_1(2^a 3^b) = \begin{cases} \left((2^{2^a} 3^{\frac{b}{2}}, \pi\left(\frac{b}{2}\right)) \right) & \text{for even } b, \\ \left(2^{2^{a+1}} 3^{\frac{b-1}{2}}, \pi\left(\frac{b-1}{2}\right) \right) & \text{for odd } b; \end{cases}$$

where $\pi(x) = 0$ for even x , $\pi(x) = 1$ for odd x , and

$$T_1(2^a 3^b) = \begin{cases} (2^a 3^{b+1}, 1) & \text{for even } b, \\ (2^a 3^{b-1}, 0) & \text{for odd } b. \end{cases}$$

There are infinitely-many operatorial statements satisfying these conditions. We shall construct three such statements.

Let r_2 be the statement defined by the algorithm

$$q_3 q_5 r, 1/\downarrow; q_2 r d_0.$$

We shall have

$$r_2(2^a 3^b 5^c) = \begin{cases} (2^{a-1} 3^b 5^c, 1) & \text{if } a > 0, \\ (2^a 3^b 5^c, 0) & \text{if } a = 0. \end{cases}$$

In the same manner we can construct the statements r_3 and r_5 .

Let p_2 be the statements defined by the algorithm 1; $r_2, 0/2$; $q_5, 1/1$; $2; r_5, 0/\downarrow$; $q_2 r_5, 0/3$; $q_2, 1/2$; $3; d_1$.

We shall have

$$p_2(2^a 3^b) = \begin{cases} (2^a 3^b, 1) & \text{for even } a, \\ (2^a 3^b, 0) & \text{for odd } a. \end{cases}$$

Starting from a state of the form $2^a 3^b$, the statement p_2 checks if a is odd. Similarly one can construct p_3 which checks if b is odd.

It follows

$$L_1 = (r_3 1 q_5)^1 (r_5 1 q_3 q_2)^1 (p_2 1 q_3) (r_2 1 q_5)^1 (r_5 1 r_5 1 q_2)^1 p_2,$$

$$R_1 = (r_3 1 q_5)^1 (r_5 1 q_2 q_2)^1 (p_3 1 q_2) (r_3 1 q_5)^1 (r_5 1 r_5 1 q_3)^1 p_3,$$

and T_1 is defined by the algorithm

$$1: p_3, 0/2; \quad r_3, 0/\downarrow; d_0; \quad 2: q_3, 1/\downarrow.$$

Theorem 4. $\mathcal{O} \leq \mathcal{M}_2$.

Proof. Let $S = \{2^a 3^b 5^c \mid a, b, c \in N\}$ be the set of states of the operatorial algorithms and let $S_1 = \{0, 1\}$ be the set of states of the Markov algorithms. The application $\alpha(2^a 3^b 5^c) = (10)^a 00(10)^b 00(10)^c$ where $(10)^n = 1010 \dots 10$ (the group 10 is n -times repeated, $(10)^0 = \Lambda$), is a injective code. The statements q_2, q_3, q_5 are α -simulated by the Markov statements $00 \rightarrow 1000, 00 \rightarrow 0010, 11 \rightarrow .00$ respectively. The statement r is α -simulated by the Markov algorithm

$$1: 100010 \rightarrow 101110; 100010 \rightarrow 00, 0/2; 1011 \rightarrow 00;$$

$$2: 101110 \rightarrow 100010, 0/\downarrow; d_0;$$

Theorem 5. $\mathcal{M}_n \leq \mathcal{R} \ (n \geq 1)$.

Proof. Let $S = \{a_0, a_1, \dots, a_{n-1}\}^*$ the set of states for the algorithms of $\mathcal{M}_n$ and let S_1 the set of states for the algorithms of $\mathcal{R}$. These states are of the form $s' = (s'_0, s'_1, s'_2, \dots)$ where $s'_0, s'_1, s'_2, \dots$ are natural numbers and all but finitely many s'_i are 0. We shall define the code $\alpha: S \rightarrow S_1$ as follows. Let $s = a_{i_0} a_{i_1}, \dots, a_{i_t}$ be a state of S . Then $(p_0^{i_0} p_1^{i_1} \dots p_t^{i_t}, x_1, x_2, \dots) \in \alpha(s)^0$ for any natural numbers $x_1, x_2, \dots$ and where p_m represents the $m + 1^{\text{th}}$ prime number. Further, for any $s = a_{i_0} a_{i_1}, \dots, a_{i_t}$ we shall use the notation $\alpha_0(s) = p_0^{i_0} p_1^{i_1} \dots p_t^{i_t}$.

One can show that exists the primitive recursive functions f and g which for any words $c, u, v \in S$ satisfy the relations

$$f(\alpha_0(c), \alpha_0(u), \alpha_0(v)) = \begin{cases} \alpha_0(T_{u,v}^{\tau}(c)) & \text{if } u \text{ is a subword in } c, \\ \alpha_0(c) & \text{if } u \text{ isn't a subword in } c; \end{cases}$$

$$g(\alpha_0(c), \alpha_0(u), \alpha_0(v)) = \begin{cases} 1 & \text{if } u \text{ is a subword in } c, \\ 0 & \text{if } u \text{ isn't a subword in } c. \end{cases}$$

It follows that exists a recursive algorithm $R_{y,z}$ so that $R_{y,z}(x, \dots) = (f(x, y, z), \dots, g(x, y, z))$ and hence for any $u, v \in S$, $T_{u,v} \leq R_{\alpha_0(u), \alpha_0(v)}(\alpha)$.

Theorem 6. $\mathcal{R} \leq \mathcal{T}_2$.

Proof. Let S and S_1 be the sets of states for the algorithms of $\mathcal{R}$ and $\mathcal{T}_2$ respectively. Let us consider the injective code defined as follows: if $s = (a_0, a_1, a_2, \dots) \in S$ then $\alpha(s) = \times 01^a_0 01^a_1 01^a_2 0 \dots$. The Turing statement C defined by the algorithm

1: $R, 1/4$; $LL, 1/3$; 2: $R, 0, 1/\downarrow$; 3: $RTL^1RT, 0, 1/\downarrow$; 4: $LL, 1/5$; $RTR^1LT, 0, 1/\downarrow$; 5: $L^1T, 0/\downarrow$; 6: $R, 0/7$; $T, 0/6, 1/\downarrow$; 7: $TR^1T, 0/\downarrow$; 8: $LTL^1L, 1/9$; $RTR^1, 0/8, 1/\downarrow$; 9: $TRTR^1LT, 1/\downarrow$; 10: $R, 1/11$; $T, 1/10, 0/\downarrow$; 11: $TL^1, 0, 1/\downarrow$;

has the property $C^{\tau} (U01^a \times 01^b 0V) = U01^b \times 01^a 0V$ for any state of the form $U01^a \times 01^b 0V \in S_1$, a and b being some natural numbers. It follows that, for any natural numbers x , the statement P_x is α -simulated by the Turing statement

$$\bar{P}_x = (R^1)^x (CL^1)^x TL (R^1 C)^x (L_1)^x d_1,$$

where $(A)^x$ means A x -times repeated, and the statement M_x is α -simulated by the statement defined by the algorithm

1: $(R^1)^x (CL^1)^x R, 1/3$; 2: $L(R^1 C)^x (L^1)^x, 0, 1/\downarrow$; 3: $TR, 0/2$;
4: $L(R^1 C)^x (L^1)^x, 0/5, 1/\downarrow$; 5: d_1 .

Theorem 7. $\mathcal{T}_n^k \leq \mathcal{R}$ ($n, k \geq 1$).

Proof. We may consider the alphabet $A = \{0, 1, \dots, n-1\}$, the number 0 being the repose symbol. Let S and S_1 be the sets of states for the algorithms of $\mathcal{T}_n^k$ and $\mathcal{R}$. We shall define the code $\alpha: S \rightarrow S_1$ as follows. For any state $s = (s_1, s_2, \dots, s_k) \in S$ where $s_i = \dots a_2^i a_1^i a_0^i \times b_0^i b_1^i b_2^i \dots$, let us consider $s' \in \alpha(s)$ for any state $s' \in S'$ of the form $s' = (x_1, y_1, x_2, y_2, \dots, x_k, y_k, \dots)$ where for $1 \leq i \leq k$, $x_i = a_0^i + na_1^i + n^2 a_2^i + \dots$ and $y_i = b_0^i + nb_1^i + n^2 b_2^i + \dots$. Then the statements L_i, R_i, T_i defined by the algorithms

$$\begin{array}{lll} t := \text{rest}(x_i, n); & t := \text{rest}(y_i, n); & y := f(y_i), \\ x_i := (x_i/n); & y_i := y_i/n; & \text{rest}(y_i, n)/\downarrow; \\ y_i := ny_i + t, t/\downarrow; & x_i := nx_i + t, b^t 1/\downarrow; & \\ a) & b) & c) \end{array}$$

where t is a number which in s' exists on the place $2k+1$ and $f(y_i) = y_i + 1$ if $b_0^i < n-1$ and $f(y_i) = nb_1^i + n^2 b_2^i + \dots$ if $b_0^i = n-1$, α -simulates the statements L_i, R_i, T_i , respectively.

It follows immediately that these algorithms are recursive.

Theorem 8. $\mathcal{O} \leq \mathcal{L}_2^1$.

Proof. Let S and S_1 be the sets of states for the two classes of algorithms. For the $\mathcal{L}_2^1$ we choose $X = \{0, 1\}$. We shall define the code $\alpha: S \rightarrow S_1$ as follows. If $s = 2^a 3^b 5^c$ then $\alpha(s) = \times 01^a 01^b 01^c 000\dots$. The statements q_2, q_3 and q_5 are α -simulated by the statements $R^1 R^1 R^1 T L^1 R T L^1 R T L L^1, R^1 R^1 R^1 T L^1 R T L L^1 L^1$ and $R^1 R^1 R^1 T L^1 L^1 L^1$ respectively and the statement r is α -simulated by the statement defined by the algorithm

1: $R, 1/2; L, 0, 1/\downarrow; 2: R^1, R, 1/3; L L^1, 0, 1/\downarrow; 3: R^1 R, 1/4; L L^1 L^1, 0, 1/\downarrow; 4: R^1 L T L^1 L T L^1 L T R T R^1 L T R T R^1 L T L L^1 L^1 d_1;$

One can immediately shown that if $p \leq q$ then $\mathcal{T}_p \leq \mathcal{T}_q$ and $\mathcal{M}_p \leq \mathcal{M}_q$.

Consequence. For any $n \geq 2$ and $k \geq 1$ the classes of algorithms $\mathcal{L}_n^k, \mathcal{T}_n^k, \mathcal{M}_n, \mathcal{O}$ and $\mathcal{R}$ are similiary.

Proof. From the Theorems 3–8 follows that $\mathcal{T}_2 \leq \mathcal{O} \leq \mathcal{M}_2, \mathcal{M}_n \leq \mathcal{R} \leq \mathcal{T}_2^1; \mathcal{T}_n^k \leq \mathcal{R}$ and $\mathcal{O} \leq \mathcal{L}_2^1$. Moreover, we have that $\mathcal{T}_2^1 \leq \mathcal{T}_n^k, \mathcal{L}_2^1 \leq \mathcal{L}_n^k \leq \mathcal{T}_n^k$ and $\mathcal{M}_2 \leq \mathcal{M}_n$. Further, the proof follows using the transitivity of the simulation relation.

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RELĂȚII DE SIMULARE ÎNTRE CLASE DE ALGORITMI

(Rezumat)

Pornind de la noțiunea generală de algoritm definită în [2] se introduc noțiunile de simulare și asemănare între algoritmi și se dau demonstrații simple pentru asemănarea claselor de algoritmi introduse în [2].

METHODS OF DATA BASE'S UP-TO-DATE PROCESSING

BY

VALENTIN HOGEA

0. Introduction. In [1], the author introduces the notion of „data base“; in [2], the chapter „Transactions on files“ describes the relation between the logical structure of a file, the determination of the material support on which he will be organized, the inner necessary memory (pre-conceived) and the way of the execution of a data base's up-to-date, without giving more details.

In this paper, the author tries to expose some of the most usual ways of up-to-date processing of the informations contained in a data base, taking care of their structure.

1. Up-to-date Processing. The up-to-date processing consists of two steps:

a) the first creation of the file (starting from a card file, a punched tape, etc.);

b) further processing (the addition of new recordings, the alteration of some recordings or the erasing of some recordings of the „old“ file).

Considering that, using the COBOL programming language, the files can be organized sequential, indexed-sequential or in random way, we can classify the following ways of up-to-date processing: a) sequential (for files on magnetic tape or disks); b) indexed-sequential (for magnetic disks); c) random (for selective files on magnetic disks).

1.1. The sequential up-to-date way. An example of such processing is given in [3] (the PROG 2 programme).

Indifferent of the dimension of file's recording, during the processing, it will be necessary to have a file of „up-to-date requests“ (generally, a punched card file).

In order to separate the requests (the transactions), the cards will have a column for „identifying“ the request, for instance: the character *A* (addition), for indicating the addition of new recordings; the character *M* (modifying), for indicating the changing of some recordings; the character *E* (erasing), for indicating the erasing of some recordings, or a numerical code with distinct numbers.

Obviously, at the first creation of the file, we shall operate only the *A* transactions, the other requests being allowed for further processing.

We must mention (for sequential files), the presence and the function of a numerical field called „identifier“ of the recording.

According with the value of this identifier, it determines the concrete position (the place) of the recording in the files. Further on, we shall denominate as „mark“, this identifier of the recording.

The up-to-date can be realized in two steps: a) the logical control (the validation) of all transactions; b) operating the transactions.

The process of validation is followed, as a rule, by the edition (at the printer's computer) of a list of wrong completed cards, and the corresponding list of error messages; only the „images“ of valid cards will be participating to the second step. The user will correct the wrong cards and will introduce them at the next processing on computer.

Briefly, the sequential up-to-date process consists of: i) the addition in the „new“ file of all recordings from the „old“ file, having the mark less than the „new“ mark from the transaction file (on *A*-type cards); ii) the addition in the „new“ file of all the *A*-type requests; iii) the modifying of some fields of the „old“ file's recordings and taking over the modified recordings into the up-to-date file; iv) the erasing of some recordings of the old file, which is done by omitting their writing into the new file (the *E*-type request, on cards).

By the programmer's point of view, these requests can be realized into two different ways: a) by three different programs (one for each type of transaction *A*, *M* or *E*); b) only by a programme with an inner sort-procedure (SORT-COBOL), which uses a parameter card.

When testing the values of the parameter card, it can be made a selection of parts of the programme which are participating to the first creation, respective, to the further processing. In order to realize the transactions, one tests the relations between the old file recording's mark and the card's mark (from the request file).

Be MARK-*N*, the mark of an *A*-type transaction and MARK-*O*, the mark of an old file recording.

If there exists the relation $\text{MARK-O} < \text{MARK-N}$, then one copies all the recordings having the mark less or equal to MARK-*O*, and creates the new recording having MARK-*N* as a mark. If we have the relation $\text{MARK-N} = \text{MARK-O}$, and we have a *M* or *E* transaction, then it will be processed the respective requests. During the up-to-date processing, all other situations are not allowed, and will be indicated by error messages (by the up-to-date programme).

Taking into account that the set of marks and the set of records (of sequential file) are in biunivocal relation, it will be eliminating the double recordings (i.e. having the same mark, but different fields), and we shall refuse (automatically) the execution of the transactions of all kinds, when having double recordings in the card file.

It is suggested that the up-to-date programme shall give an up-to-date (at the computer's printer), in which all the done transactions should be mentioned.

1.2. The indexed-sequential up-to-date process. An indexed-sequential file consists of two areas, named the main area and the overflow area.

The main area contains the recordings created at the first processing of the file, and the index table connected to the file. The overflow area contains the addition recordings further on created, in up-to-date processings.

Unlike the sequential up-to-date, in which, a new recording is placed between two existing recordings (of marks in strictly increasing order), at the indexed-sequential up-to-date process, a new recording is placed in the overflow area and it realizes a „table“ of chaining, which will precise the exact place where the recording should have been placed.

This, gives the conclusion, that at all transactions, one is consulting the recording's mark, the index tables and the list of recording's chaining. We must mention that, at the first creation of the indexed-sequential file, the marks shall be ordered into a strictly increasing order.

It's reasonable to use first a programme for creating a sequential file (using the sequential up-to-date), which can be used like input file into the programme who will create the first indexed-sequential file.

Taking into account that an indexed-sequential file can be used also in direct (random) access, the transactions will become very simple on operating: the addition will be realized into the overflow area; when having equal marks, it will be operated the M or E request (a logical one, by invalidating to the further reading) of the recordings.

The drawback of the indexed-sequential up-to-date process is the big time of access to the recording, due to the fact, that when finding it, we must be consulting the index table and the chaining lists.

The benefit consists in the fact, that the programs using such a way, have an unsophisticated procedure.

1.3. The random (direct) up-to-date processing. One of the difficult problems of selective organized files (direct access mode) is the statement of the random access formula (a calculus formula which shows the dependence between the actual key and the symbolic key of the recording). An example of such a formula is given in [1], another in [2], pp. 205—206, 2nd volume.

The creation of a selective file is made in two steps: a) firstly one creates a sequential file (only with A -type transactions), whose marks are in increasing order; b) one creates the first selective file (only with A -type transactions), using the random chosen process.

The further on processing it will be made in direct access, using the same random formula, like in b) step: i) for additions, one is checking if the calculated value of the key is different to all the keys of the old file; in a suitable case, one creates the new recording; ii) for modifyings (or erasings), at the coincidence of calculated key, the requested transaction will be done (on the same material support).

The direct up-to-date processing programs have the benefit of fluency, clarity and simplicity.

2. A Comparison Between Up-to-date Methods. In [1] it is described a virtual structure of a data base. The data base's up-to-date processing consists of making all the transactions of its compounding files, in the same time with the checking of logical and physique relations wich exists between the fields or the complete recordings of these files.

For the technological operations, technological dimensions, stocks, contracts, partners in contracts, personel and payment files, we suggest the direct up-to-date processing. For the „article“ files, the structure of

the finite products, we advice the direct access or the indexed-sequential up-to-date process.

In order to increase the efficiency of the computing time, we suggest that, in the application programs (using a data base) not to use sequential files, but their indexed-sequential or selective variants, which will be exploited in direct access.

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METODE DE ACTUALIZARE A INFORMAȚIILOR UNEI BAZE DE DATE

(Rezumat)

Autorul prezintă câteva din cele mai utilizate metode de actualizare a informațiilor unei baze de date, ținând cont de tipul fișierelor conținute: secvențiale, secvențial-indexate și organizate în acces direct (random).

O CLASĂ DE CURBE BERTRAND

DE

AL. NICOLESCU

1. Definesc această clasă prin ecuațiile intrinseci

$$(B) \quad \frac{1}{\rho} = \frac{a_1 s + b_1}{as + b}, \quad \frac{1}{\tau} = \frac{a_2 s + b_2}{as + b},$$

unde am notat cu ρ — raza de curbura, τ — raza de torsiune a curbei (B) ; adică curbura și torsiunea sînt funcții omografice de arcul s al curbei.

2. Pentru a arăta că, în adevăr, aceste curbe sînt curbe Bertrand, vom căuta să determinăm constantele α și β astfel ca să avem

$$\frac{\alpha}{\beta} + \frac{\beta}{\tau} = 1,$$

(sau să eliminăm arcul s între ecuațiile (B)); este suficient să luăm

$$a_1 \alpha + a_2 \beta = a,$$

$$b_1 \alpha + b_2 \beta = b,$$

obținînd pentru α și β soluții bine determinate, atîta vreme cît $D = a_1 b_2 - a_2 b_1 \neq 0$. Vom avea atunci

$$\begin{aligned} D\alpha &= ab_2 - a_2 b = D_1 & \rightarrow & \frac{D_1}{\rho} + \frac{D_2}{\tau} = D. \\ D\beta &= ba_1 - ab_1 = D_2 \end{aligned}$$

3. Înainte de a trece mai departe, vom cerceta cazurile de excepție: $D = 0$ și $D_1, D_2 = 0$.

În primul caz, avem $a_2/a_1 = b_2/b_1 = k$, deci $1/\tau = k(1/\rho)$ și curba (B) se reduce la o *elice*.

În cazul $D_1 = 0$, rezultă $1/\tau = \text{const.}$, iar dacă $D_2 = 0$, avem $1/\rho = \text{const.}$; deci curbe (B) cu *torsiunea sau curbura constantă*.

4. Excluzînd aceste cazuri, conchidem că curbele (B) formează o clasă de curbe *Bertrand*, depinzînd de 5 (cinci) parametri. Le vom numi *curbe (B) omografice*, indicate prin $B-O$.

Să considerăm transformatele afine ale curbelor $B-O$, adică definite prin

$$\frac{1}{\rho_1} = \frac{A_1}{\rho} + \frac{B_1}{\tau} + C_1, \quad \frac{1}{\tau_1} = \frac{A_2}{\rho} + \frac{B_2}{\tau} + C_2,$$

A_i, B_i, C_i constante date. Se vede imediat, ținând seama de expresiile lui ρ și τ , că $1/\rho_1$ și $1/\tau_1$ sînt funcții omografice de arcul s . Printre transformatele afine ale curbelor $B-O$ sînt :

- a) *elici*, dacă $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$;
 b) *curbe cu curbura constantă*, dacă $\frac{A_1}{D_1} = \frac{B_1}{D_2}$;
 c) *curbe cu torsiune constantă*, dacă $\frac{A_2}{D_1} = \frac{B_2}{D_2}$.

5. Un caz interesant se obține luînd $a_1 = 0$, deci

$$\frac{1}{\rho} = \frac{b_1}{as + b}, \quad \frac{1}{\tau} = \frac{a_2s + b_2}{as + b},$$

de unde

$$\frac{\rho}{\tau} = \frac{a_2s + b_2}{b_1} = As + B.$$

Această relație caracterizează *curbele Enneper*; așa dar printre curbele $B-O$ sînt și curbe Enneper, care — în cazul de față — au proprietatea că raza de curbura este o funcție liniară de arcul curbei (sau proporțională cu arcul corespunzător).

6. Să arătăm acum corelația între curbele $B-O$ și *curbele Tîțeica sferice*.

Vom considera curba (C) din spațiu raportată la triedrul ei Frenet $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ și corespondența ei (S), definită prin

$$(1) \quad \mathbf{r}_0 = \mathbf{t} \sin \omega + \mathbf{b} \cos \omega \quad (\omega = * \text{ const.}).$$

Se constată, imediat că $\mathbf{r}_0^2 = 1$, deci (S) este o curbă sferică, pe sfera unitară. Dacă notăm cu s și σ arcele respective ale celor două curbe obținem prin derivare

$$\frac{d\sigma}{ds} \mathbf{t}_0 = \frac{\mathbf{n}}{\rho} \sin \omega + \frac{\mathbf{n}}{\tau} \cos \omega = \mathbf{n}h,$$

$$h = \frac{\sin \omega}{\rho} + \frac{\cos \omega}{\sigma}$$

și de aici

$$(2) \quad \frac{d\sigma}{ds} = h.$$

Să presupunem acum că (C) este o curbă Bertrand, definită de relația $(\sin \omega)/\rho + (\cos \omega)/\tau = h$, h fiind o constantă reală; deducem $\mathbf{t}_0 = \mathbf{n}$, deci corespondența $(C) \leftrightarrow (S)$ este caracterizată de paralelismul dintre tangenta la (S) și normala principală la (C) .

Într-o serie de lucrări anterioare am definit curbele sferice prin ecuația

$$(V) \quad \dot{\mathbf{r}}_0 = v(\sigma) \mathbf{r}_0 \times \ddot{\mathbf{r}}_0;$$

ținând seama acum de relația (1) și înlocuind, obținem

$$\mathbf{n} = \frac{-v(\sigma)}{h} \left(\frac{\cos \omega}{\rho} - \frac{\sin \omega}{\tau} \right) \mathbf{n},$$

adică

$$-\frac{\cos \omega}{\rho} + \frac{\sin \omega}{\tau} = \frac{h}{v(\sigma)};$$

alăturînd și ecuația ce definește curbele Bertrand

$$\frac{\sin \omega}{\rho} + \frac{\cos \omega}{\tau} = h,$$

găsim în definitiv

$$\frac{1}{\rho} = h \sin \omega - \frac{h \cos \omega}{v(\sigma)}, \quad \frac{1}{\tau} = h \cos \omega + \frac{h \sin \omega}{v(\sigma)},$$

ecuațiile unei curbe Bertrand în corespondență cu o curbă sferică (V) , corespondența fiind definită prin relația (1).

Tot în lucrările respective am arătat că pentru curbele Țițeica sferice avem $v(\sigma) = a\sigma + b$ și în acest caz rezultă

$$\frac{1}{\rho} = \frac{a_1\sigma + b_1}{a\sigma + b}, \quad \frac{1}{\tau} = \frac{a_2\sigma + b_1}{a\sigma + b},$$

adică tocmai curbele $B - O$, definite la începutul acestei lucrări. Arcul σ putem să-l înlocuim cu s , potrivit formulei (2).

7. Rezultatele stabilite aici sugerează cercetarea următoarelor chestiuni de geometrie diferențială clasică:

1. problema unei corespondențe între curbele $B - O$ și conicele planului, observînd că ambele curbe depind de 5 parametri esențiali;

2. problema folosirii acestor curbe în teoria aplicabilității suprafețelor, în generarea suprafețelor minima, în deformarea cuadricelor de rotație, așa cum a arătat Bertrand G a m b i e r, în celebrul său memoriu publicat în 1926, în analele Universității din Lille.

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Universitatea din București,
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UNE CLASSE DE COURBES BERTRAND

(Résumé)

On considère dans l'espace euclidien S_3 les courbes définies par les équations intrinsèques (B) et on montre qu'elles sont des courbes de Bertrand.

COMPLÉMENTS À L'ÉTUDE DIFFÉRENTIELLE DES VARIÉTÉS DE QUADRIQUES

PAR

DAVID RIMER

1. Introduction. Notons par $V(a; C; S)$ les variétés avec a paramètres ($a = 1, 2$ ou 3), générées par une quadrique C , dans un espace S , S étant un espace projectif, affine ou euclidien à trois dimensions (espaces notés par P_3, A_3 ou F_3).

Dans [1] — [4] et autres sept Notes y citées, nous avons considéré une variété générique $V(a; C; S)$ et, en utilisant la méthode du repère mobile de Cartan, nous avons obtenu le système complet d'invariants, le repère Frenet, les formules Frenet et une configuration géométrique invariante, à l'aide de laquelle nous avons donné des interprétations aux éléments du repère et aux invariants. Notons par $F(a; C; S)$ et $G(a; C; S)$ les formules Frenet, respectivement la configuration géométrique construite. Puisque l'algorithme de la méthode du repère mobile implique certaines restrictions (l'indépendance des formes de Pfaff choisies pour formes principales de la variété, l'annulation d'autres formes de Pfaff, certaines valeurs pour diverses fonctions etc.), a été délimitée une classe de variétés $V(a; C; S)$, pour lesquelles les résultats obtenus sont valables. Il s'impose alors la suivante

Définition. Nommons variétés $V(a; C; S)$ d'espèce M , les variétés pour lesquelles, dans le cas générique a été déterminé le repère Frenet et l'on a donné des interprétations géométriques aux éléments de ce repère. Les autres variétés $V(a; C; S)$ seront nommées d'espèce N .

Au fur et à mesure que certaines des variétés qui ont été exceptées (variétés d'espèce N), seront étudiées complètement, elles deviendront variétés d'espèce M . Ceci impliquera à introduire un indice dans la notation : $V_i(a; C; S), F_i(a; C; S), G_i(a; C; S), i = 1, 2, \dots$. D'ailleurs, une situation analogue l'on a, par exemple, pour les courbes W planes en P_3 , où l'on distingue courbes W de classe I, ou II.

Dans ce qui suit, nous notons le repère fondamental de l'espace par $\{B_\lambda, V\}$ ou $\{O; \bar{i}_k\}$ et le repère Frenet de la variété par $\{A_\alpha, U\}$ ou $\{M, \bar{I}_j\}$, ($\alpha, \lambda = 0, 1, 2, 3, ; j, k = 1, 2, 3$), selon que $S = P_3$, respectivement $S = A_3$ ou E_3 . Rapportée au repère fondamental, la variété aura pour équation

$$(1) \quad C = a_{\alpha\beta} y^\alpha y^\beta = 0, \quad a_{\alpha\beta} = a_{\beta\alpha},$$

où $a_{\alpha\beta}$ sont fonctions régulières C^h ($h \geq 1$) des paramètres principaux de la variété et, rapportée au repère Frenet, l'équation de la variété aura la forme canonique

$$(2) \quad C = \varepsilon_\alpha (x^\alpha)^2 = 0, \quad (\varepsilon_\alpha = -1, 0 \text{ ou } +1).$$

Dans cette Note, nous complétons l'étude faite dans les Notes citées, en solutionnant les problèmes suivants, posés par le prof. Dan I. P a p u c, auquel nous remercions de même pour les suggestions données.

Étant donnée une variété $V(a; C; S)$ par une équation de la forme (1), avec $a_{\alpha\beta}$ précisés, il faut déterminer l'espèce de la variété.

2) Étant connu qu'une variété $V(a; C; S)$ donnée par une équation de la forme (1), est d'espèce M , il faut déterminer les valeurs de ses invariants.

3) Étant donnée une variété $V(a; C; S)$ d'espèce M , par les valeurs de ses invariants, il faut déterminer l'équation de la forme (1), de cette variété.

Mentionnons que la solution de ces problèmes permet à utiliser la théorie construite dans la solution de divers exercices applicatifs.

2. Détermination de l'espèce d'une variété. La réponse à ce problème est donnée par le

T h é o r è m e 1. Les propositions suivantes sont équivalentes :

(α) La variété $V(a; C; S)$ donnée par une équation de la forme (1), avec des coefficients $a_{\alpha\beta}$ précisés, est d'espèce M .

(β) Ayant l'équation de la forme (1) d'une variété $V(a; C; S)$, on peut reproduire intégralement l'algorithme de la méthode du repère mobile utilisé au cas de la variété générique d'espèce M , avec les mêmes restrictions, jusqu'on fixe tous les paramètres secondaires et l'on obtient les formules Frenet.

(γ) Ayant l'équation de la forme (1) d'une variété $V(a; C; S)$, on peut construire la configuration $G(a; C; S)$ obtenue au cas de la variété générique $V(a; C; S)$ d'espèce M .

D é m o n s t r a t i o n. (α) $\Rightarrow$ (β). La variété particulière considérée, étant d'espèce M , il s'ensuit, selon la définition, qu'on peut lui appliquer intégralement l'algorithme poursuivi au cas de la variété générique respective.

(β) $\Rightarrow$ (γ). La construction de la configuration $G(a; C; S)$ est immédiate, en poursuivant seulement la description de cette configuration donnée au cas de la variété générique. Elle ne réclame que l'équation de la forme (1) de la variété et les formules Frenet.

(γ) $\Rightarrow$ (α). En ayant la configuration $G(a; C; S)$ et l'en poursuivant l'interprétation des éléments du repère Frenet donnée au cas de la variété générique $V(a; C; S)$, on obtient le repère Frenet de la variété particulière considérée. Ainsi, au cas $S = P_3$, on obtient des relations de la forme

$$(3) \quad A_\alpha = m_\alpha^\lambda B_\lambda$$

et au cas $S = A_3$ ou $S = E_3$, on a

$$(3') \quad \overline{M} = a^k \overline{i}_k, \quad \overline{I}_j = a_j^k \overline{i}_k,$$

où m_α^λ , a^λ et a_j^λ sont fonctions régulières C^h ($h \geq 1$) des paramètres principaux de la variété, les matrices (m_α^λ) , (a_j^λ) sont non dégénérées et, de plus, si $S = E_3$, (a_j^λ) est une matrice orthogonale. Vu que, pour la variété particulière considérée nous avons déterminé la repère Frenet et la configuration $G(a; C; S)$ qui donne des interprétations géométriques aux éléments de ce repère, la variété est d'espèce M .

Exemple I. En notant par E , une quadrique non dégénérée de type elliptique en P_3 , soit la variété $V(1; E; P_3)$ dont l'équation de la forme (1) est

$$(I.1) \quad E = -\frac{1}{a}(y^0)^2 + \frac{1}{s^2+2}(y^1)^2 + \frac{1}{s+1}(y^2)^2 + \frac{1}{s}(y^3)^2 = 0, \quad s > 0.$$

Tentons de reproduire la voie suivie en [1], pp. 60–63, au cas de la variété générique $V(1; E; P_3)$. En notant par $\{\bar{A}_\alpha, \bar{U}\}$ les repères d'ordre zéro (correspondent au sousgroupe de stabilité de la quadrique), l'équation de la variété, rapportée à ces repères est ([1], rel. (3.11)),

$$(I.2) \quad E = -(\bar{x}^0)^2 + (\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2 = 0.$$

Il s'ensuit que les coordonnées y_α^λ et $\bar{x}^\alpha$ sont liées par des relations de la forme

$$y^0 = \pm \sqrt{s} \bar{x}^0, \quad y^1 = \pm \sqrt{s^2+2} \bar{x}^1, \quad y^2 = \pm \sqrt{s+1} \bar{x}^2, \quad y^3 = \pm \sqrt{s} \bar{x}^3, \\ (\beta, \gamma, \delta \in \{1, 2, 3\}, \quad \delta \neq \beta \neq \gamma \neq \delta). \dagger$$

Ils résultent alors les relations suivantes entre $\bar{A}_\alpha$ et B_λ :

$$(I.3) \quad \bar{A}_0 = \pm \sqrt{s} B_0, \quad \bar{A}_1 = \pm \sqrt{s^2+2} B_\beta, \quad \bar{A}_2 = \pm \sqrt{s+1} B_\gamma, \\ \bar{A}_3 = \pm \sqrt{s} B_\delta,$$

c'est-à-dire que les sommets $\bar{A}_\alpha$ sont fixes, puisqu'ils coïncident aux sommets fixes B_λ .

De (I.3), il résultent par différentiation les relations

$$d\bar{A}_\alpha = \theta_\alpha \bar{A}_\alpha \quad (\alpha = \overline{0, 3}),$$

où θ_α sont des formes de Pfaff. En les identifiant avec les relations génériques (2.1) de [1], $d\bar{A}_\alpha = \omega_\alpha^\beta \bar{A}_\beta$, il en résulte $\omega_0^0 = \omega_1^0 = 0$ et, donc, $\omega_0^0 - \omega_1^0 = 0$, ce qui est contraire à la condition imposée au cas de la variété générique, $\omega_0^0 - \omega_1^0 \neq 0$, (puisque nous l'avons défini comme pfaffien principal de la variété). Nous concluons que pour la variété (I.1) considérée, la proposition (β) du théorème 1 n'est pas satisfaite et donc (I.1) n'est pas une variété d'espèce M .

On obtient la même conclusion en tentant de construire la configuration $G(1; E; P_3)$ décrite en [1], pp. 63–64.

Exemple II. La variété $V(1; E; P_3)$ d'équation

$$(II.1) \quad E \equiv \left(-\frac{15}{16} e^{-6s} + \frac{1}{16} e^{2s} - \frac{1}{8} e^{-2s} \right) (y^0)^2 + e^{2s} (y^1)^2 + e^{2s} (1 + 4s^2) (y^2)^2 + \\ + e^{2s} (y^3)^2 + \frac{1}{2} (e^{-2s} - e^{2s}) y^0 y^1 - 4s e^{2s} y^2 y^3 = 0, \quad s \in (-1, 1),$$

est d'espèce M . En effet, nous construisons la configuration $G(1; E; P_3)$ qui consiste d'une certaine quadrique non dégénérée réglée et quatre cônes quadratiques ([1], pp. 63 — 64) dont les équations s'obtiennent en partant de l'équation (II.1). En poursuivant l'interprétation géométrique du repère Frenet (ibid. p. 64), on obtient les sommets du repère Frenet

$$(II.2) \quad \begin{aligned} A_0 &= \theta_0 \left(e^{2s} B_0 + \frac{1}{4} (e^{2s} - e^{-2s}) B_1 \right), & A_1 &= \theta_1 B_1, \\ A_2 &= \theta_2 (B_2 + 2s B_3), & A_3 &= \theta_3 B_3, \end{aligned}$$

où θ_α sont des facteurs de proportionnalité qu'il faut déterminer. Avec

$U = \sum_{\alpha=0}^3 A_\alpha$, on a les repères $\{A_\alpha, U\}$ qui dépendent de s et de θ . Les relations (II.2) définissent une transformation de coordonnées

$$\begin{aligned} y^0 &= \theta_0 e^{2s} x^0, & y^1 &= \frac{1}{4} \theta_0 (e^{2s} - e^{-2s}) x^0 + \theta_1 x^1, \\ y^2 &= \theta_2 x^2, & y^3 &= 2 \theta_2 s x^2 + \theta_3 x^3. \end{aligned}$$

En introduisant ces expressions de y^λ dans l'équation (II.1), on obtient l'équation de la variété considérée, rapportée au repère (II.2). En identifiant cette équation, avec l'équation générique $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0$, il résulte $\theta_0 = \pm e^s$, $\theta_1 = \pm e^{-s}$, $\theta_2 = \pm e^{-s}$, $\theta_3 = \pm e^{-s}$. Nous introduisons ces valeurs de θ_α dans l'équation de la variété rapportée au repère (II.2). Puis nous fixons au repère l'orientation positive en imposant les conditions $A_\alpha(0) = B_\alpha$, (la valeur $s = 0$ est permise, puisque $s \in (-1, 1)$). On obtient ainsi

$$(II.3) \quad \begin{aligned} A_0 &= e^{3s} B_0 + \frac{1}{4} (e^{3s} - e^{-s}) B_1, & A_1 &= e^{-s} B_1, \\ A_2 &= e^{-s} B_2 + 2s e^{-s} B_3, & A_3 &= e^{-s} B_3. \end{aligned}$$

3. Détermination des valeurs des invariants d'une variété donnée d'espèce M . Étant donnée une variété $V(a; C; S)$ d'espèce M , on peut obtenir les relations (3) ou (3'). Au cas $S = P_3$, en différenciant les relations (3), on obtient

$$(4) \quad dA_\alpha = dm_\alpha^\lambda B_\lambda.$$

De (3) il résulte $B_\lambda = \tilde{m}_\lambda^\beta A_\beta$, avec $m_\alpha^\lambda \tilde{m}_\lambda^\beta = \delta_\alpha^\beta$. Alors les relations (4) deviennent

$$(5) \quad dA_\alpha = dm_\alpha^\lambda \tilde{m}_\lambda^\beta A_\beta.$$

En identifiant les relations (5) avec les formules Frenet $F(a; C; S)$ établies au cas de la variété générique respective d'espèce M , on obtient les valeurs des invariants de la variété donnée. Au cas $S = A_3$ ou $S = E_3$ on procède d'une manière analogue, en différenciant les relations (3').

Exemple III. Soit la variété $V(1; E; P_3)$ d'espèce M donnée par l'équation (II.1). En dérivant les relations (II.3) et en remplaçant dans les relations obtenues les sommets B_λ par leurs valeurs, résultées de (II.3), on obtient les formules Frenet $F(1; E; P_3)$ de la variété (II.1) considérée

$$(III.1) \quad \frac{dA_0}{ds} = 3A_0 + A_1, \quad \frac{dA_1}{ds} = -A_1, \quad \frac{dA_2}{ds} = -A_2 + A_3, \quad \frac{dA_3}{ds} = -A_3.$$

En identifiant (III.1), avec les formules (3.14) de [1], (où $\varepsilon = +1$), c'est-à-dire avec les formules Frenet de la variété générique $V(1; E; P_3)$, on obtient les valeurs des invariants de la variété donnée (II.1)

$$(III.2) \quad p_1 = -1, \quad p_2 = 2, \quad p_3 = 1, \quad p_4 = p_5 = p_6 = p_7 = 0, \quad p_8 = 2.$$

4. Détermination de l'équation d'une variété d'espèce M , donnée par les valeurs de ses invariants. a) *Le cas $S = P_3$.* Dans les formules Frenet $F(a; C; P_3)$ établies pour la variété générique, nous remplaçons les invariants par leurs valeurs. Les relations obtenues constituent un système d'équations différentielles (si $a = 1$), ou un système d'équations aux différentielles totales (si $a = 2$ ou 3), ayant A_α comme fonctions inconnues. En intégrant ce système et en particulierisant les constantes d'intégration à l'aide de certaines conditions initiales, on obtient la solution particulière

$$(6) \quad A_\alpha = q_\alpha^\lambda B_\lambda, \quad \det(q_\alpha^\lambda) \neq 0,$$

où q_α^λ sont fonctions des paramètres principaux de la variété donnée. Avec

$$U = \sum_{\alpha=0}^3 A_\alpha, \quad \text{nous avons le repère Frenet } \{A_\alpha, U\} \text{ de la variété considérée.}$$

Des relations (6), il résulte que, entre les coordonnées x^α et y^λ d'un point M de P_3 , il y a les relations

$$(7) \quad y^\lambda = q_\alpha^\lambda x^\alpha, \quad x^\alpha = \tilde{q}_\mu^\alpha y^\mu \quad (q_\alpha^\lambda \tilde{q}_\lambda^\beta = \delta_\alpha^\beta).$$

Au cas de la variété générique $V(a; C; S)$, le repère Frenet a été obtenu dans l'hypothèse que, rapportée au repère Frenet, la variété a l'équation (2). Avec (7), l'équation (2) de la variété devient

$$(8) \quad C = \varepsilon_\alpha (\tilde{q}_\mu^\alpha y^\mu)^2 = 0.$$

b) *Les cas $S = A_3$, $S = E_3$.* En procédant de la même manière, on obtient, dans les conditions initiales choisies,

$$(6') \quad \bar{M} = b^k \bar{i}_k, \quad \bar{I}_j = b_j^k \bar{i}_k,$$

avec la matrice (b_j^k) non dégénérée et, de plus, orthogonale si $S = E_3$. D'ici ils résultent les relations

$$(7') \quad y^j = b^j + b_k^j x^k, \quad x^k = \tilde{b}_j^k (y^j - a^j).$$

Avec (7.2), l'équation (2) devient

$$(8') \quad C = \varepsilon_k [\tilde{b}_j^k (y^j - a^j)]^2 + \varepsilon_0 = 0.$$

Les équations (8) et (8') caractérisent la variété donnée $V(a; C; S)$ d'espèce M , rapportée au repère fondamental de l'espace.

Exemple IV. Déterminons l'équation de la forme (1) d'une variété $V(1; E; P_3)$ d'espèce M , dont les invariants sont donnés par (III.2).

Avec ces valeurs des invariants, les formules $F(1; E; P_3)$ de [1] deviennent (III.1). En intégrant ce système, dans les conditions initiales $A_\alpha(0) = B_\alpha$, on obtient le repère Frenet (II.3).

D'ici ils résultent les relations entre les coordonnées x^α et y^λ . En les introduisant dans l'équation (2) de la variété, on obtient l'équation de la forme (1) de la variété considérée. Elle est justement (II.1).

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COMPLEMENTE LA STUDIUL DIFERENȚIAL AL VARIETĂȚILOR DE CUADRICE

(Rezumat)

Se definesc varietățile de speța M de cuadrice și se dă o teoremă de caracterizare analitică și geometrică a acestor varietăți. Se rezolvă problema: fiind dată o varietate de speța M , prin ecuația ei raportată la reperul fundamental, să i se găsească invariantii și reciproc, să se determine ecuația varietății când sînt dați invariantii.

F-QUASI-CONNECTIONS

BY

ELENA VAMANU

1. Introduction. Let M be a C^∞ -differentiable, n -dimensional manifold with an almost product structure defined by the tensor field F_j^i . In [4] there are determined all the quasi-connections $\bar{\Gamma}_j^i$ which F_j^i is covariant constant. These quasi-connections were called F -quasi-connections. They are given by

$$(1) \quad \bar{\Gamma}_{kh}^i = F_k^m \left(\gamma_{mh}^i + \frac{1}{2} F_j^i \nabla_m F_h^j \right) + W_{hk}^i,$$

where γ_{mh}^i is an arbitrary but fixed connection on the manifold M , ∇_j is the operator of covariant differentiation with respect to γ_{mh}^i , and W_{hk}^i is an arbitrary tensor field that satisfies the condition

$$(2) \quad Q_{mh}^{ki} W_{kj}^h = W_{mj}^i,$$

where Q_{mh}^{ki} denotes the Obata operator defined by

$$(3) \quad Q_{mh}^{ki} = \frac{1}{2} (\delta_m^k \delta_h^i + F_k^m F_h^i).$$

If Γ_{kh}^i is the F -quasi-connection given by

$$(4) \quad \Gamma_{kh}^i = F_k^m \left(\gamma_{mh}^i + \frac{1}{2} F_j^i \nabla_m F_h^j \right),$$

then (1) may be written in the following form (see [4] for details)

$$(5) \quad \bar{\Gamma}_{kh}^i = \Gamma_{kh}^i + W_{hk}^i.$$

Now, we consider the transformation $\Gamma_{kh}^i \rightarrow \bar{\Gamma}_{kh}^i$ defined by (5), W_{hk}^i being an arbitrary tensor field that satisfies the conditions (2).

It is easy to see that the set of these transformations is a group G isomorphic to the additive group of the tensor fields W . For certain choices of W we obtain certain subgroups of the group G . In the present paper we aim to determine two subgroups of the group G and to find some of their invariants.

2. Special F -quasi-connections. 2.1. Let us take

$$(6) \quad W_{hk}^i = \delta_h^i p_k,$$

where p_k is an arbitrary covector field.

One verifies easily that W defined by (6) satisfies (2) and hence we obtain the transformation $\Gamma \rightarrow \bar{\Gamma}$ given by

$$(7) \quad \bar{\Gamma}_{kh}^i = \Gamma_{kh}^i = \delta_h^i \rho_k.$$

The set of these transformations defines the subgroup G_1 of the group G . We shall now determine some invariants of the group G_1 .

Let $\Gamma_{jk}^i, \bar{\Gamma}_{jk}^i$ be two F -quasi-connections of type (7) and $T_{jk}^i, \bar{T}_{jk}^i$ be their torsion tensors, and $R_{jkh}^i, \bar{R}_{jkh}^i$ their curvature tensors.

Using (7) and the expressions of these tensors given in [2], we obtain

$$(8) \quad \bar{T}_{jk}^i = T_{jk}^i + \delta_k^i \rho_j - \delta_j^i \rho_k,$$

$$(9) \quad \bar{R}_{hij}^k = R_{hij}^k + \delta_h^k (F_i^m \rho_{j,m} - F_j^m \rho_{i,m} + \rho_i F_{m,n}^t F_{ij}^n F_j^m).$$

By contracting i, k in (8) we get

$$(10) \quad \bar{T}_{ji}^i = T_{ji}^i + (n-1)\rho_j.$$

If we put

$$(11) \quad \bar{T}_j = \bar{T}_{ji}^i, \quad T_j = T_{ji}^i,$$

from (10) we have

$$(12) \quad \rho_j = \frac{\bar{T}_j - T_j}{n-1}.$$

Substituting (12) in (8) we obtain

$$(13) \quad \bar{T}_{jk}^i - \frac{1}{n-1} \delta_k^i \bar{T}_j + \frac{1}{n-1} \delta_j^i \bar{T}_k = T_{jk}^i - \frac{1}{n-1} \delta_k^i T_j + \frac{1}{n-1} \delta_j^i T_k.$$

Thus we find the first invariant tensor of the group G_1 given by

$$(14) \quad I_{1jk}^i = T_{jk}^i - \frac{1}{n-1} \delta_k^i T_j + \frac{1}{n-1} \delta_j^i T_k.$$

By contracting k, h , in (9) we obtain

$$(15) \quad \bar{R}_{kij}^k = R_{kij}^k + n(F_i^m \rho_{j,m} - F_j^m \rho_{i,m} + \rho_i F_{m,n}^t F_{ij}^n F_j^m).$$

If we denote

$$(16) \quad \bar{P}_{ij} = \bar{R}_{kij}^k, \quad P_{ij} = R_{kij}^k$$

we obtain from (15)

$$(17) \quad F_i^m \rho_{j,m} - F_j^m \rho_{i,m} + \rho_i F_{m,n}^t F_{ij}^n F_j^m = \frac{\bar{P}_{ij} - P_{ij}}{n}.$$

Substituting (17) in the right hand side of (9) we have

$$(18) \quad \bar{R}_{hij}^k - \frac{1}{n} \delta_h^k \bar{P}_{ij} = R_{hij}^k - \frac{1}{n} \delta_h^k P_{ij}.$$

Thus, we find the second invariant tensor of the group G_1 , given by

$$(19) \quad I_{2,ij}^k = R_{hij}^k - \frac{1}{n} \delta_h^k P_{ij}.$$

Now, if we contract k, j in (9) we get

$$(20) \quad \bar{R}_{hi} = R_{h,i} + F_i^m p_{h,m} - F_h^m p_{i,m} + p_i F_{m,n}^t F_{[i}^n F_{h]}^m,$$

where

$$(21) \quad \bar{R}_{ht} = \bar{R}_{htk}, \quad R_{ht} = R_{htk}.$$

Now, from (9) and using (20) we obtain

$$(22) \quad \bar{R}_{hij}^k - \delta_h^k \bar{R}_{jt} = R_{hij}^k - \delta_h^k R_{jt}.$$

We have thus obtained the third invariant tensor of the group G_1 given by

$$(23) \quad I_{3hij}^k = R_{hij}^k - \delta_h^k R_{jt}.$$

We can obtain another invariant else of the group G_1 but this is not a tensor. Indeed, if we contract i, h in (7) we have

$$(24) \quad \bar{\Gamma}_k = \Gamma_k + n p_k,$$

where

$$(25) \quad \bar{\Gamma}_k = \bar{\Gamma}_{kt}^i, \quad \Gamma_k = \Gamma_{kt}^i.$$

It immediately follows from (7), by use of (24)

$$(26) \quad \bar{\Gamma}_{kh}^i - \frac{1}{n} \delta_h^i \bar{\Gamma}_k = \Gamma_{kh}^i - \frac{1}{n} \delta_h^i \Gamma_k.$$

Hence, we obtain the invariant

$$(27) \quad \Omega_{kh}^i = \Gamma_{kh}^i - \frac{1}{n} \delta_h^i \Gamma_k.$$

2.2. Now, we take

$$(28) \quad W_{hk}^i = F_h^i p_k,$$

with p_k an arbitrary covector field.

One can easily verify that W defined by (28) satisfies (2) and hence we obtain the transformation $\Gamma \rightarrow \bar{\Gamma}$ given by

$$(29) \quad \bar{\Gamma}_{kh}^i = \Gamma_{kh}^i + F_h^i p_k.$$

The set of these transformations defines the subgroup G_2 of the group G .

Proceeding in the same way as in 2.1, we can obtain some invariants of G_2 .

Let $T_{jk}^i, \bar{T}_{jk}^i$ be the torsion tensors, and $R_{kij}^h, \bar{R}_{kij}^h$ the curvature tensors with respect to $\Gamma_{jk}^i, \bar{\Gamma}_{jk}^i$ respectively. Then we have

$$(30) \quad \bar{T}_{jk}^i = T_{jk}^i + F_k^i p_j - F_j^i p_k,$$

$$(31) \quad \bar{R}_{hij}^k = R_{hij}^k + F_h^k (F_i^m p_{j,m} - F_j^m p_{i,m} + p_i F_{m,n}^t F_{[i}^n F_{j]}^m).$$

Multiplying (30) by F_i^m , we get

$$(32) \quad \bar{T}_{jk}^i F_j^m = T_{jk}^i F_j^m + \delta_k^m p_j - \delta_j^m p_k.$$

If we contract m, k in (32), we obtain

$$(33) \quad \bar{T}_{jk}^i F_j^k = T_{jk}^i F_j^k + (n-1)p_j$$

from which it follows

$$(34) \quad p_j = \frac{(\bar{T}_{jk}^i - T_{jk}^i) F_j^k}{n-1}.$$

Substituting (34) in (30) we have

$$(35) \quad \begin{aligned} & \bar{T}_{jk}^i - \frac{1}{n-1} F_k^i \bar{T}_{jm}^h F_h^m + \frac{1}{n-1} F_j^i \bar{T}_{km}^h F_h^m = \\ & = T_{jk}^i - \frac{1}{n-1} \left[F_k^i T_{jm}^h F_h^m + \frac{1}{n-1} F_j^i T_{km}^h F_h^m \right]. \end{aligned}$$

We have thus obtained the first invariant tensor of the group G_2 , given by

$$(36) \quad J_{1jk}^i = T_{jk}^i + \frac{1}{n-1} F_h^m (F_j^i T_{km}^h - F_k^i T_{jm}^h).$$

Now, multiplying (31) by F_i^h , we obtain

$$(37) \quad R_{hij}^k F_i^h = R_{hij}^k F_i^h + \delta_i^k (F_h^m p_{j,m} - F_j^m p_{i,m} + p_i F_{m,n}^t F_{[i}^n F_{j]}^m).$$

If we contract k, l in (37) we have

$$(38) \quad \bar{R}_{hij}^k F_i^h = R_{hij}^k F_i^h + n(F_h^m p_{j,m} - F_j^m p_{i,m} + p_i F_{m,n}^t F_{[i}^n F_{j]}^m).$$

From (31) by means of (38), one obtains

$$(39) \quad \bar{R}_{hij}^k - \frac{1}{n} F_h^k \bar{R}_{nij}^m F_m^n = R_{hij}^k - \frac{1}{n} F_h^k R_{nij}^m F_m^n.$$

We thus find the second invariant tensor of the group G_2 given by

$$(40) \quad J_{2hij}^k = R_{hij}^k - \frac{1}{n} F_h^k R_{nij}^m F_m^n.$$

Now, contracting k, j in (31) and using (21), we obtain

$$(41) \quad \bar{R}_{hi} = R_{hi} + F_h^k (F_i^m p_{k,m} - F_k^m p_{i,m} + p_i F_{m,n}^t F_{[i}^n F_{k]}^m)$$

and multiplying (41) by F_j^h , we have

$$(42) \quad \bar{R}_{hi} F_j^h = R_{hi} F_j^h + F_i^m p_{j,m} - F_j^m p_{i,m} + p_i F_{m,n}^t F_{[i}^n F_{j]}^m.$$

From (31) and (42) it follows

$$(43) \quad \bar{R}_{hij}^k - F_h^k \bar{R}_{mi} F_j^m = R_{hij}^k - F_h^k R_{mi} F_j^m.$$

We have thus obtained the third invariant tensor of the group G_2 given by

$$(44) \quad J_{ghij}^k = R_{hij}^k - F_h^k R_{mi} F_j^m.$$

Finally we can find a new invariant of the group G_2 but this is not a tensor.

Multiplying (29) by F_m^h we have

$$(45) \quad \bar{\Gamma}_{kh}^i F_m^h = \Gamma_{ih}^i F_m^h + \delta_m^i p_k.$$

If we contract i, m in (45) we obtain

$$(46) \quad p_k = \frac{(\bar{\Gamma}_{kh}^m - \Gamma_{kh}^m) F_m^h}{n}.$$

From (29) it follows, by use of (46),

$$\bar{\Gamma}_{kh}^i - \frac{1}{n} F_h^i \bar{\Gamma}_{im}^i F_l^m = \Gamma_{ih}^i - \frac{1}{n} F_h^i \Gamma_{km}^i F_l^m.$$

Hence, we find the invariant

$$(47) \quad \Omega_{1kh}^i = \Gamma_{kh}^i - \frac{1}{n} F_h^i \Gamma_{im}^i F_l^m.$$

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F-CVASI-CONEXIUNI

(Rezumat)

Data o varietate cu structură aproape produs, mulțimea tuturor cvasi-conexiunilor în raport cu care tensorul structurii aproape produs este covariant constant, poate fi dotată cu o structură de grup. În lucrare se consideră două subgrupuri ale acestui grup și se determină o serie de invarianți ai lor.

$$(41) \quad \frac{1}{\lambda} \frac{d\lambda}{dt} = \frac{1}{\lambda} \frac{d\lambda}{dt} - \frac{1}{\lambda} \frac{d\lambda}{dt}$$

Finally, we can find a new invariant of the group, but this is not a

Multiplying (39) by $\frac{1}{\lambda}$ we have

$$(42) \quad \frac{1}{\lambda} \frac{d\lambda}{dt} = \frac{1}{\lambda} \frac{d\lambda}{dt} + \frac{1}{\lambda} \frac{d\lambda}{dt}$$

If we contract (42) with (41) we obtain

$$(43) \quad \frac{1}{\lambda} \frac{d\lambda}{dt} = \frac{1}{\lambda} \frac{d\lambda}{dt} - \frac{1}{\lambda} \frac{d\lambda}{dt}$$

From (39) it follows, by use of (43),

$$\frac{1}{\lambda} \frac{d\lambda}{dt} = \frac{1}{\lambda} \frac{d\lambda}{dt} - \frac{1}{\lambda} \frac{d\lambda}{dt}$$

Hence, we find the invariant

$$(44) \quad \frac{1}{\lambda} \frac{d\lambda}{dt} = \frac{1}{\lambda} \frac{d\lambda}{dt} - \frac{1}{\lambda} \frac{d\lambda}{dt}$$

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A-CVAST-CONNECTION

(Received)

Il est à noter que la structure de groupe, l'ensemble des transformations de la forme $x \rightarrow ax + b$, est un sous-groupe de la structure de groupe. En fait, la structure de groupe est un sous-groupe de la structure de groupe.

KAWAGUCHI CONNECTIONS ASSOCIATED TO A FINSLER CONFORMAL ALMOST SIMPLECTICAL STRUCTURE

BY

I. GHINEA

Recently [7], prof. R. Miron studied the class of Finsler connections $F\tilde{\Gamma}$ having the property that Kawaguchi canonical metric connection $K\tilde{\Gamma}$ coincides with Cartan connection $C\tilde{\Gamma}$.

Applying prof. R. Miron's method, we consider in [2] the class of Finsler connections with the property that Kawaguchi conformal connection coincides with the canonical conformal connection $F\tilde{\Gamma}^a$ of $g_{ij}(x, y)$ and further, in [3], the class of Finsler connections with the property that Kawaguchi almost simplectical connection $K\tilde{\Gamma}^a$ coincides with the canonical almost simplectical connection $F\tilde{\Gamma}^a$ of $a_{ij}(x, y)$.

In this paper we are dealing with the class of Finsler connections having the property that Kawaguchi conformal almost simplectical connection $K\tilde{\Gamma}^a$ coincides with the canonical, conformal almost simplectical connection $F\tilde{\Gamma}^{Ka}$ of $a_{ij}(x, y)$.

1. Let M be a $2n$ -dimensional, differentiable Finsler manifold. A Finsler conformal almost simplectical structure on M is defined by means of a pseudo-tensor field $a_{ij}(x, y)$ (y being the support element in point x), with the properties:

- i) $a_{ij}(x, y)$ is defined up to a local positive and arbitrary factor $\rho(x, y)$;
- ii) $a_{ij}(x, y) = -a_{ji}(x, y)$;
- iii) $\det(a_{ij}(x, y)) \neq 0, y \neq 0$.

Let associate to $a_{ij}(x, y)$ the operators

$$(1.1) \quad \theta_{kh}^{ij} = \frac{1}{2} (\delta_k^i \delta_h^j - a_{kh} a^{ij}), \quad \theta_{kh}^{*ij} = \frac{1}{2} (\delta_k^i \delta_h^j + a_{kh} a^{ij}).$$

A Finsler connection $F\tilde{\Gamma} = (N, F, C)$ is called compatible with the conformal almost simplectical structure ρa_{ij} if

$$(1.2) \quad \begin{cases} a_{ij|k} = 2\omega_k a_{ij}, \\ a_{ij|k} = 2\lambda_k a_{ij}, \end{cases}$$

take place, where $\omega_k = \omega_k(x, y)$ and $\lambda_k = \lambda_k(x, y)$ are homogeneous covariant vectors in y of degree 0 and respective -1 .

In the paper [1] we prove the

Theorem 1.1. *All the Finsler conformal almost symplectical connections are given by*

$$(1.3) \quad \begin{cases} N_j^h = \dot{N}_j^h + X_j^h, \\ F_{jk}^h = \dot{F}_{jk}^h + \frac{1}{2} a^{ph} \left(a_{pj|k} - 2\omega_k a_{pj} - X_k^q \frac{\partial a_{pj}}{\partial y_q} \right) + \epsilon_{jq}^{ph} U_{pk}^q, \\ C_{jk}^h = \dot{C}_{jk}^h + \frac{1}{2} a^{ph} (a_{pj|k} - 2\lambda_k a_{pj}) + \epsilon_{jq}^{ph} V_{pk}^q, \end{cases}$$

where $F\dot{\Gamma} = (\dot{N}_j^h, \dot{F}_{jk}^h, \dot{C}_{jk}^h)$ is a fixed Finsler connection, $| \cdot |$ are the covariant h -derivative and v -derivative with respect to $F\dot{\Gamma}$ and $X_j^h, U_{pk}^q, V_{pk}^q$ are arbitrary fields of Finsler tensors.

Since U_{pk}^q and V_{pk}^q are arbitrary, we pose

$$\begin{cases} U_{pk}^q = -2\delta_k^q \omega_p + U_{pk}^q, \\ V_{pk}^q = -2\delta_k^q \lambda_p + V_{pk}^q \end{cases}$$

and then (1.3) become

$$(1.3') \quad \begin{cases} N_j^h = \dot{N}_j^h + X_j^h, \\ F_{jk}^h = \dot{F}_{jk}^h + \frac{1}{2} a^{ph} a_{pj|k} - \delta_j^h \omega_k - \delta_k^h \omega_j + a_{jl} a^{ph} \omega_p - \frac{1}{2} X_k^q a^{ph} \frac{\partial a_{pj}}{\partial y_q} + \epsilon_{jq}^{ph} U_{pk}^q, \\ C_{jk}^h = \dot{C}_{jk}^h + \frac{1}{2} a^{ph} a_{pj|k} - \delta_j^h \lambda_k - \delta_k^h \lambda_j + a_{jl} a^{ph} \lambda_p + \epsilon_{jq}^{ph} V_{pk}^q. \end{cases}$$

Among the set of conformal almost symplectical connections (1.3) and (1.3') we report two important.

The first one is the connection obtained from (1.3') if we consider as given connection $F\dot{\Gamma} = (\dot{N}, \dot{F}, \dot{C})$ introduced in [3] and $X_j^h = 0, U_{pk}^q = 0, V_{pk}^q = 0$. We are calling these connection canonical, conformal almost symplectical. Denoted by $F\dot{\Gamma} = (\dot{N}, \dot{F}, \dot{C})$, such a connection is given by

$$(1.4) \quad \begin{cases} \overset{ca}{N}_j^h = \overset{a}{N}_j^h, \\ \overset{ca}{F}_{jk}^h = \overset{a}{F}_{jk}^h - \delta_j^h \omega_k - \delta_k^h \omega_j + a_{jl} a^{h1} \omega_1, \\ \overset{ca}{C}_{jk}^h = \overset{a}{C}_{jk}^h - \delta_j^h \lambda_k - \delta_k^h \lambda_j + a_{jl} a^{h1} \lambda_1. \end{cases}$$

The second important conformal almost symplectical connection can be obtained from (1.3) for $X_j^h = 0, U_{pk}^q = 0, V_{pk}^q = 0$. Since it generalizes Kawaguchi connection for the conformal almost symplectical case, it will be called Kawaguchi conformal almost symplectical connection associated with

Finsler connection $F\dot{\Gamma}$ and denoted by $\overset{cK}{K}\dot{\Gamma} = (\overset{cK}{N}, \overset{cK}{F}, \overset{cK}{C})$:

$$(1.5) \quad \begin{cases} \overset{cK}{N}_j^h = \overset{\circ}{N}_j^h, \\ \overset{cK}{F}_{jk}^h = \overset{\circ}{F}_{jk}^h + \frac{1}{2} a^{ph} (a_{pj|k} - 2\omega_k a_{pj}), \\ \overset{cK}{C}_{jk}^h = \overset{\circ}{C}_{jk}^h + \frac{1}{2} a^{ph} (a_{pj|k} - 2\lambda_k a_{pj}). \end{cases}$$

We now turn to the following question: obtain all Finsler connections $F\overset{\circ}{\Gamma}$ with the property that Kawaguchi conformal, almost symplectical connection $\overset{c}{K}\overset{\circ}{\Gamma}$ coincides with the canonical, conformal almost symplectical connection $F\overset{ca}{\Gamma}$. In this purpose we shall prove.

Theorem 1.2. *All connections $F\overset{\circ}{\Gamma}$ with the property $\overset{c}{K}\overset{\circ}{\Gamma} = F\overset{ca}{\Gamma}$ are given by*

$$(1.6) \quad \begin{cases} \overset{\circ}{N}_j^h = \overset{q}{N}_j^h, \\ \overset{\circ}{F}_{jk}^h = F_{jk}^h - \delta_j^h \omega_k - \delta_k^h \omega_j + a_{jk} a^{hl} \omega_l + \epsilon_{mj}^{*hp} X_{pk}^m = \overset{ca}{F}_{jk}^h + \epsilon_{mj}^{*hp} X_{pk}^m, \\ \overset{\circ}{C}_k = \overset{q}{C}_k - \delta_j^h \lambda_k - \delta_k^h \lambda_j + a_{jk} a^{hl} \lambda_l + \epsilon_{mj}^{*hp} Y_{pk}^m = \overset{ca}{C}_k + \epsilon_{mj}^{*hp} Y_{pk}^m, \end{cases}$$

where $\overset{q}{C}\overset{\circ}{\Gamma} = (\overset{q}{N}, \overset{q}{F}, \overset{q}{C})$ is the canonical almost symplectical connection, X_{pk}^m, Y_{pk}^m are arbitrary Finsler tensor fields and θ^* is the second operator associated to $a_{ij}(x, y)$.

Proof. Let be $(A_j^h, E_{jk}^h, D_{jk}^h)$ the tensors obtained as difference of the pair $(F\overset{\circ}{\Gamma}, F\overset{ca}{\Gamma})$:

$$(1.7) \quad \begin{cases} \overset{\circ}{N}_j^h = \overset{ca}{N}_j^h + A_j^h, \\ \overset{\circ}{F}_{jk}^h = \overset{ca}{F}_{jk}^h + E_{jk}^h, \\ \overset{\circ}{C}_{jk}^h = \overset{ca}{C}_{jk}^h + D_{jk}^h. \end{cases}$$

Taking account of (1.7) in (1.5) and having also in mind (1.2) we get

$$(1.8) \quad \begin{cases} \overset{cK}{N}_j^h = \overset{ca}{N}_j^h + A_j^h, \\ \overset{cK}{F}_{jk}^h = \overset{ca}{F}_{jk}^h - \frac{1}{2} a^{ph} A_k^m \frac{\partial a_{pj}}{\partial y^m} + \theta_{jm}^{ph} E_{pk}^m, \\ \overset{cK}{C}_{jk}^h = \overset{ca}{C}_{jk}^h + \theta_{jm}^{ph} D_{pk}^m. \end{cases}$$

By imposing the condition $\overset{c}{K}\overset{\circ}{\Gamma} = F\overset{ca}{\Gamma}$, Eqs. (1.8) give

$$(1.9) \quad A_j^h = 0, \quad A_{jm}^{ph} E_{pk}^m = 0, \quad \theta_{jm}^{ph} D_{pk}^m = 0.$$

The general solution of system (1.9) is

$$(1.10) \quad A_j^h = 0, \quad B_{jk}^h = \epsilon_{hj}^{*im} X_{mk}^h, \quad D_{jk}^h = \epsilon_{hj}^{*im} Y_{mk}^h,$$

where X_{mk}^h and Y_{mk}^h are arbitrary tensor fields.

Let now introduce (1.10) in (1.7); taking account also of (1.4) we obtain (1.6).

2. Let be

$$\mathcal{K} = \{F\overset{\circ}{\Gamma} \mid \overset{c}{K}\overset{\circ}{\Gamma} = F\overset{c}{\Gamma}\}.$$

About Finsler connections $F\overset{\circ}{\Gamma}$ from $\mathcal{K}$ we shall prove the following theorems.

Theorem 2.1. *In the set of $F\overset{\circ}{\Gamma}$ connections from $\mathcal{K}$ there exists a single conformal, almost symplectical connection, namely the canonical, conformal almost symplectical connection $F\overset{c}{\Gamma}$.*

Proof. Imposing the condition that the connection $F\overset{\circ}{\Gamma}$ from (1.6) be conformal almost symplectical i.e. satisfy (1.2), and taking into account that the canonical connection $F\overset{c}{\Gamma}$ is conformal almost symplectical, we get the following relations:

$$(2.1) \quad \begin{cases} (a_{hj}(\overset{*}{h}p_{mi} + a_{ih}e_{mj}^{\overset{*}{h}p}))X_{pk}^m = 0, \\ (a_{hj}(\overset{*}{h}p_{mi} + a_{ih}e_{mj}^{\overset{*}{h}p}))Y_{pk}^m = 0, \end{cases}$$

equivalent to

$$(2.2) \quad \theta_{mi}^{\overset{*}{h}p}X_{pk}^m = 0, \quad \theta_{mi}^{\overset{*}{h}p}Y_{pk}^m = 0.$$

Introducing (2.2) in (1.6) we get $F\overset{\circ}{\Gamma} = F\overset{c}{\Gamma}$.

Theorem 2.2. *All the connections $F\overset{\circ}{\Gamma}$ from $\mathcal{K}$ present the following properties regarding the $(h)h$ -torsion T and the $(v)v$ -torsion S :*

$$(2.3) \quad \begin{cases} a_{ih}\overset{\circ}{T}_{jk}^h + a_{jh}\overset{\circ}{T}_{ki}^h + a_{ih}\overset{\circ}{T}_{ij}^h = 0, \\ a_{ih}\overset{\circ}{S}_{jk}^h + a_{jh}\overset{\circ}{S}_{ki}^h + a_{ih}\overset{\circ}{S}_{ij}^h = 0. \end{cases}$$

The proof results from a direct computation of the left hand terms in (2.3), taking into account also (1.6).

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CONEXIUNI KAWAGUCHI ASOCIATE UNEI STRUCTURI FINSLER
CONFORM APROAPE SIMPLECTICE

(Rezumat)

Se determină clasa de conexiuni Finsler care au proprietatea că conexiunea conform aproape simplctică Kawaguchi coincide cu conexiunea conform aproape simplctică canonică.

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CONEXIONES ASOCIADAS A ESTRUCTURAS DE FINSLER CONFORME APROXIMACIÓN SIMPLICITICA

(Resumen)

Se determinan las clases de conexiones de Finsler en las que las propiedades de conexiones conformes a la aproximación simpléctica coinciden con las propiedades de conexiones conformes a la aproximación simpléctica.

ON THE INTERPOLATION OF 2^n BANACH SPACES

I. THE LIONS-PEETRE INTERPOLATION METHOD

BY

DICESAR LASS FERNANDEZ

0. Introduction. The extensions of the now classical interpolation methods, either real or complex, from the context of two Banach spaces to several have been made by several authors. The Lions-Peetre method [9] was extended for $n+1$ spaces and n parameters by Yoshikawa [12]. The extension of the K and J method of J. Peetre for $n+1$ spaces and n parameters was given by G. Sparr [11] who showed that the interpolation spaces generated by his methods coincide with the ones generated by Yoshikawa's methods. On the other hand Fernandez [1] gives an extension of the K and J method of a different kind: it concerns with 2^n spaces and n parameters. The goal of this note is to give an extension of the Lions-Peetre interpolation method for 2^n spaces and n parameters and to show that this method, under certain conditions, generate the same space as the K and J method.

For another real interpolation method of 2^n Banach spaces and n parameters see Fernandez [8]. For extensions of the complex method see Lions [8], Favini [3], Fernandez [5], Bertolo [2] and Fernandez-Bertolo [7].

Applications of the results of this Note to the so called Mangeron's equation will appear in a forthcoming paper.

Notations. Let us denote by $\square$ the set of the $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ such that $k_j = 0$ or 1 ($j = 1, \dots, n$). When $n = 1$ and $n = 2$ we shall have $\square = \{0, 1\}$ and $\square = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, respectively. The indices k of the objects we shall concern will run always in $\square$.

Given $\xi_0 = (\xi_0^1, \dots, \xi_0^n)$ and $\xi_1 = (\xi_1^1, \dots, \xi_1^n)$ in $\mathbb{R}^n$, we define

$$\underline{\xi} = (\xi_k = (\xi_{k_1}^1, \dots, \xi_{k_n}^n) \mid k = (k_1, \dots, k_n) \in \square).$$

Also, if $P_0 = (p_0^1, \dots, p_0^n)$ and $P_1 = (p_1^1, \dots, p_1^n)$ are n -tuples with $1 \leq p_0^j, p_1^j \leq \infty$ ($j = 1, \dots, n$) we define

$$\mathbf{P} = (P_k = (p_{k_1}^1, \dots, p_{k_n}^n) \mid k = (k_1, \dots, k_n) \in \square).$$

When $n = 2$, we shall have

$$\underline{\xi} = ((\xi_0^1, \xi_0^2), (\xi_1^1, \xi_0^2), (\xi_0^1, \xi_1^2), (\xi_1^1, \xi_1^2))$$

$$\mathbf{P} = ((p_0^1, p_0^2), (p_1^1, p_0^2), (p_0^1, p_1^2), (p_1^1, p_1^2)).$$

For L^p spaces, with mixed norms, see Benedek-Panzone [1].

1. The Spaces $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$. **1.1.** We shall introduce in this section the interpolation space $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$. To so, we shall give an extension of the Lions-Peetre interpolation method for 2^n Banach spaces.

Let $\mathbf{E} = (E_k \mid k \in \square)$ be 2^n Banach spaces continuously embedded in a same linear Hausdorff space V .

We shall consider the linear hull $\Sigma \mathbf{E}$ and the intersection $\cap \mathbf{E}$ (when $n = 2$ we shall have $\Sigma \mathbf{E} = E_{00} + E_{10} + E_{01} + E_{11}$ and $\cap \mathbf{E} = E_{00} \cap E_{10} \cap E_{01} \cap E_{11}$), endowed with the norms

$$\|x\|_{\Sigma \mathbf{E}} = \inf \{ \sum_k \|x_k\|_{E_k} \mid x = \sum_k x_k, \quad x_k \in E_k, \quad k \in \square \}$$

and

$$\|x\|_{\cap \mathbf{E}} = \max \{ \|x\|_{E_k} \mid k \in \square \}$$

respectively. The spaces $\Sigma \mathbf{E}$ and $\cap \mathbf{E}$ are Banach spaces continuously embedded in V (see [4]).

1.2. The linear space of all functions $u: \mathbf{R}^n \rightarrow \Sigma \mathbf{E}$ such that

$$(1.2.1) \quad e^{\xi_k t} u \in L^{p_k}(E_k), \quad k \in \square,$$

will be denoted by $W(\mathbf{P}; \underline{\xi}; \mathbf{E})$. The space $W(\mathbf{P}; \underline{\xi}; \mathbf{E})$ is a Banach space under the norm

$$(1.2.2) \quad \|u\|_{W(\mathbf{P}; \underline{\xi}; \mathbf{E})} = \max_k \|e^{\xi_k t} u\|_{L^{p_k}(E_k)}.$$

Let us set the following hypothesis (H) on $\underline{\xi} = (\xi_k \mid k \in \square)$:

$$(H) \quad \xi_0^j \xi_1^j < 0, \quad j = 1, \dots, n.$$

1.3. If $\underline{\xi} = (\xi_k \mid k \in \square)$ satisfies (H) and $u \in W(\mathbf{P}; \underline{\xi}; \mathbf{E})$ then

$$(1.3.1) \quad \int_{\mathbf{R}^n} \|u(t)\|_{\Sigma \mathbf{E}} dt < \infty.$$

1.4. The space of $x \in \Sigma \mathbf{E}$ for which there exist $u \in W(\mathbf{P}; \underline{\xi}; \mathbf{E})$ such that

$$(1.4.1) \quad x = \int_{\mathbf{R}^n} u(t) dt$$

will be denoted by $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$. Under the norm

$$(1.4.2) \quad \|x\|_{S(\mathbf{P}; \underline{\xi}; \mathbf{E})} = \inf \left\{ \|u\|_{W(\mathbf{P}; \underline{\xi}; \mathbf{E})} \mid x = \int_{\mathbf{R}^2} u(t) dt \right\}$$

the space $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$ become a Banach space.

1.5. Proposition. For all $x \in \cap \mathbf{E}$ there is a constant $C > 0$ such that

$$(1.5.1) \quad \|x\|_{S(\mathbf{P}; \underline{\xi}; \mathbf{E})} \leq C \|x\|_{\cap \mathbf{E}},$$

that is

$$(1.5.2) \quad \cap \mathbf{E} \subset S(\mathbf{P}; \underline{\xi}; \mathbf{E}).$$

Proof. Let $\varphi \in C_c^\infty(\mathbf{R}^n)$ with $\int \varphi = 1$. Now, if $x \in \cap \mathbf{E}$ and putting $u(t) = \varphi(t)x$, we have $x = \int u$ and then the result follows.

We are going to show that the spaces $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$ depend only of the parameter $\theta = (\theta^1, \dots, \theta^n)$ and of $\mathbf{P}$. First we have

1.6. Proposition. Let $A = (\lambda_1, \dots, \lambda_n)$ be given with $\lambda_j \neq 0$. Then

$$(1.6.1) \quad S(\mathbf{P}; \underline{\xi}; \mathbf{E}) = S(\mathbf{P}; \lambda \underline{\xi}; \mathbf{E})$$

with equivalent norms, where

$$\xi \Lambda = (\lambda \xi_k = (\lambda_1 \xi_{k_1}, \dots, \lambda_n \xi_{k_n}) \mid k = (k_1, \dots, k_n) \in \square).$$

Proof. Let $u \in W(\mathbf{P}; \underline{\xi}; \mathbf{E})$ and define u_Λ by

$$u_\Lambda(t) = u_{\lambda_1, \dots, \lambda_n}(t_1, \dots, t_n) = \lambda_1, \dots, \lambda_n u(\lambda_1 t_1, \dots, \lambda_n t_n)$$

then $u_\Lambda \in W(\mathbf{P}; \underline{\xi}; \mathbf{E})$ and

$$\|e^{\lambda \xi_k} u_\Lambda\|_{L^{P_k}(\mathbf{E}_k)} = C(\Lambda; k) \|e^{\xi_k} u\|_{L^{P_k}(\mathbf{E}_k)}.$$

Since $\int u_\Lambda dt = \int u(t) dt$ it follows from (1.4.2) that

$$\|x\|_{S(\mathbf{P}; \underline{\xi}; \mathbf{E})} = C(\Lambda) \|x\|_{S(\mathbf{P}; \lambda \underline{\xi}; \mathbf{E})}$$

as desired.

1.7. Proposition. If $\Theta = \{\theta - k \mid \theta = (\theta^1, \dots, \theta^n), k = (k_1, \dots, k^n) \in \square\}$, we have

$$(1.7.1) \quad S(\mathbf{P}; \underline{\xi}; \mathbf{E}) = S(\mathbf{P}; \Theta; \mathbf{E})$$

and the norms are equivalents.

Proof. If we take $\lambda^j = 1/(\xi_0^j - \xi_1^j)$ and put $\theta^j = \xi^j(\xi_0^j - \xi_1^j)$, $j = 1, 2, \dots, n$, in (1.6.1) the result follows.

We don't now if the number of parameters can be reduced further as in Peetre [10].

Next, we state the interpolation property of the spaces $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$.

1.8. Proposition. Let $E = (E_k \mid k \in \square)$ and $F = (F_k \mid k \in \square)$ be two families of Banach spaces embedded in the linear Hausdorff spaces V and W , respectively. If T is a linear operator from ΣE into ΣF such that $T|_{E_k}$ is a bounded linear operator from ΣE_k into ΣF_k , $k \in \square$, then for all ξ and P the linear operator

$$T: S(\mathbf{P}; \underline{\xi}; \mathbf{E}) \rightarrow S(\mathbf{P}; \underline{\xi}; \mathbf{F})$$

is bounded.

Proof. Let $x \in S(\mathbf{P}; \underline{\xi}; \mathbf{E})$ and $u = u(t)$ as in (1.4.1). Since T is bounded from ΣE into ΣF we have $Tx = T(\int u(t) dt) = T(\int (u(t)) dt)$.

Thus, if M_k is the norm of $T: E_k \rightarrow F_k$, we have

$$\|Tx\|_{\theta; Q; J} \leq t^{-\theta} J(t; T(u(t))) \|_{L^Q} \leq (\max_k M_k) \|t^{-\theta} J(t; u(t))\|_{L^Q}$$

and from this inequality the result follows.

2. The Spaces $\mathcal{S}(\mathbf{P}; \underline{\xi}; \mathbf{E})$. As in Lions-Peetre [9] we can introduce the interpolation space $\mathcal{S}(\mathbf{P}; \underline{\xi}; \mathbf{E})$.

2.1. Let $(u_k \mid k \in \square)$ be a family of function $u_k: \mathbf{R}^n \rightarrow \Sigma \mathbf{E}$ such that

$$(2.1.1) \quad e^{\xi_k t_k} u_k \in L^{P_k}(E_k) \quad (k \in \square)$$

and

$$(2.1.2) \quad \Sigma_k v_k(t) = \text{const. a. e.}$$

We define the space $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$ as the linear space of $x \in \Sigma \mathbf{E}$ for which there is functions u_k , $k \in \square$, so that (2.1.1) is verified and such that

$$(2.1.3) \quad x = \sum_k v_k(t) \quad \text{a. e.}$$

Under the norm

$$\|x\|_{S(\mathbf{P}; \underline{\xi}; \mathbf{E})} = \inf \{ \max_k \|e^{\xi_k t} v_k\|_{L^p_k(\mathbf{E}_k)} \mid x = \sum_k v_k(t) \}$$

the linear space $\mathcal{S}(\mathbf{P}; \underline{\xi}; \mathbf{E})$ becomes a Banach space.

The spaces $\mathcal{S}(\mathbf{P}; \underline{\xi}; \mathbf{E})$ have properties all analogous to the spaces $S(\mathbf{P}; \underline{\xi}; \mathbf{E})$. Moreover, as we shall see later, they coincide.

3. The Spaces $(\mathbf{E}; \Theta; P; K)$ and $(\mathbf{E}; \theta; P; J)$. We recall briefly the definitions of the spaces $(\mathbf{E}; \theta; P; K)$ and $(\mathbf{E}; \xi; P; J)$ as given in Fernandez [4].

Let us consider the following functional norms in $\mathbf{R}_+^n = \mathbf{R}_+ \times \dots \times \mathbf{R}_+$.

$$K(t; x) = \inf \{ \sum_k t^k \|x_k\|_{\mathbf{E}_k} \mid x = \sum_k x_k \}$$

and

$$J(t; x) = \max \{ t^k \|x\|_{\mathbf{E}_k} \mid k \in \square \}$$

(here $t^k = t^{k_1} \dots, t^{k_n}$).

Now, let L_*^P stands for a L^P space with mixed norm, respect to the measure $d^*t = dt_1/t_1 \dots dt_n/t_n$.

3.1. If $0 < \theta = (\theta^1, \dots, \theta^n) < 1$ and $1 \leq P = (p^1, \dots, p^n) \leq \infty$, we define $(\mathbf{E}; \theta; P; K)$ to be the space of all elements $x \in \Sigma \mathbf{E}$ such that

$$(3.1.1) \quad t^{-\theta} K(t; x) \in L_*^P$$

and $(\mathbf{E}; \theta; P; J)$ to be the space of all elements $x \in \Sigma \mathbf{E}$ for which there exists a $u \in L_1(\mathbf{R}_+^n; \Sigma \mathbf{E})$ with $u(\mathbf{R}_+^n) \subset \cap \mathbf{E}$ which satisfy

$$(3.1.2) \quad x = \int_{\mathbf{R}_+^n} u(t) d^*t$$

and

$$(3.1.3) \quad t^{-\theta} J(t; u(t)) \in L_*^P.$$

The spaces $(\mathbf{E}; \theta; P; K)$ and $(\mathbf{E}; \theta; P; J)$ are Banach spaces under the norms

$$\|x\|_{\theta; P; K} = \|t^{-\theta} K(t; x)\|_{L_*^P}$$

and

$$\|x\|_{\theta; P; J} = \inf \left\{ \|t^{-\theta} J(t; u)\|_{L_*^P} \mid x = \int_{\mathbf{R}_+^n} u(t) d^*t \right\}$$

respectively.

The following Proposition is proved in [4].

3.2. Proposition. If $0 < \theta = (\theta^1, \dots, \theta^n) \leq 1$ and $1 \leq P = (p^1, \dots, p^n) \leq \infty$ we have

$$(\mathbf{E}; \xi; P; K) = (\mathbf{E}; \xi; P; J)$$

with equivalent norms.

4. The Equivalence of the Definitions. 4.1. Proposition. The following conditions are equivalent:

(i) There exists a $u \in L_+^1(\Sigma \mathbf{E})$, with $u(\mathbf{R}_+^n) \in \cap \mathbf{E}$, such that

$$(4.1.1) \quad \begin{cases} x = \int_{\mathbf{R}_+^n} u(t) d^*t, \\ t^{-\theta} J(t; u(t)) \in L_*^P. \end{cases}$$

(ii) There exists a $v \in L^1(\mathbf{R}^n, \Sigma \mathbf{E})$, with $v(\mathbf{R}^n) \in \cap \mathbf{E}$, such that

$$(4.1.2) \quad \begin{cases} x = \int_{\mathbf{R}_+^n} v(s) ds \\ e^{(k-\theta)s} v(s) \in L^P(E_k), \quad k \in \square. \end{cases}$$

(iii) There exists a 2^n -tuple $(v_k | k \in \square)$ of $\Sigma \mathbf{E}$ -valued on $\mathbf{R}^n$ such that

$$(4.1.3) \quad \begin{cases} x = \sum_k u_k(s), \quad \text{a.a. } s \in \mathbf{R}^n, \\ e^{(k-\theta)s} u_k(s) \in L^P(E_k), \quad k \in \square. \end{cases}$$

(iv) There exists elements $x_k \in E_k$, $k \in \square$ such that

$$(4.1.4) \quad \begin{cases} x = \sum_k x_k, \\ t^{-\theta} K(t; x) \in L_*^P. \end{cases}$$

Proof. (i) implies (ii). Let x and u be as in (i) and such that (4.1.1) holds. The change of variables $s_j = \log t_j$, $j = 1, \dots, n$ implies that there exists a $v(s) = v(s_1, \dots, s_n)$ such that

$$x = \int_{\mathbf{R}^n} v(s) ds.$$

On the other hand

$$\begin{aligned} \|e^{(k-\theta)s} v(s)\|_{L^P(E_k)} &\leq \|\max_k \|e^{(k-\theta)s} v(s)\|_{E_k}\|_{L^P} = \|\max_k \|t^{k-\theta} u(t)\|_{E_k}\|_{L_*^P} = \\ &= \|t^{-\theta} \max_k t^k \|u(t)\|_{E_k}\|_{L_*^P} = \|t^{-\theta} \max_k t^k \|u(t)\|_{E_k}\|_{L_*^P}, \end{aligned}$$

and (4.1.2) follows.

(ii) implies (iii). Let v be as in (ii) and such that (4.1.2) holds. Now, let us put χ_k , $k \in \square$, the characteristic functions of the sets $(-\infty, 0) \times (-\infty, 0)$, $(-\infty, 0) \times (0, \infty)$, $(0, \infty) \times (-\infty, 0)$ and $(0, \infty) \times (0, \infty)$ (in this point we are giving the proof only for $n = 2$).

Then

$$I = \sum_k \chi_k$$

and

$$x = I * u = \sum_k \chi_k * u = \sum_k v_k.$$

Now, since $\exp[(k - \theta)t] \chi_k \in L^1$, $k \in \square$, we have

$$e^{(k-\theta)t} v_k = (e^{(k-\theta)t} \chi_k) * (e^{(k-\theta)t} u) \in L^p_k(E_k)$$

$k \in \square$, and (4.1.3) follows.

(iii) implies (iv). Let $(v_k | k \in \square)$ be given as in (iii) and such that (4.1.3) holds. By a change of variables we get from (4.1.3) that

$$x = \sum_k v_k(t) \text{ (a.a.t.)}, \quad \text{and} \quad t^{-\theta} \{ \sum_k t^k \| u_k(t) \|_{E_k} \} \in L^p_*$$

and from this we get finally

$$x = \sum_k x_k \quad \text{and} \quad t^{-\theta} K(t; x) \in L^p_*.$$

(iv) implies (i). This is proposition 3.2.

The proof of equivalence of the norms of the spaces $\mathcal{S}(\mathbf{P}; \Theta; \mathbf{E})$, $S(\mathbf{P}; \Theta; \mathbf{E})$, $(\mathbf{E}; \Theta; \mathbf{P}; \mathbf{K})$ and $(\mathbf{E}; \Theta; \mathbf{P}; \mathbf{J})$ is implicit in the above proof. We are done.

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ASUPRA INTERPOLĂRII SPAȚIILOR 2^n BANACH

I. Metoda de interpolare Lions-Peetre

(Rezumat)

Se prezintă o extensiune a metodei de interpolare Lions-Peetre la 2^n spații Banach și n parametri și se arată că această metodă generează, sub anumite condiții, același spațiu ca și metodele K și J ale lui J. Peetre.

POSITIVE SOLUTIONS FOR SOME FUNCTIONAL EQUATIONS

BY

ADRIAN CORDUNEANU

1. The projective metric ideas, used by P.J. Bushell ([1], [2]) in the study of certain non-linear integral Volterra and Fredholm equations, may be applied without major modifications in the case of same equations on a infinite interval. Denote by $C[0, \infty)$ the Banach space of all continuous bounded functions on $[0, \infty)$ with the norm $\|x\| = \sup_{t \geq 0} |x(t)|$. Also denote by $C_g[0, \infty)$ the Banach space defined as follows

$$(1) \quad C_g[0, \infty) = \left\{ x; x = x(t) \text{ continuous for } t \geq 0, \|x\|_g = \sup_{t \geq 0} \frac{|x(t)|}{g(t)} < \infty \right\}$$

where $g = g(t)$ is a fixed continuous function for $t \geq 0$, with $g(t) > 0$ for $t > 0$. The case $g(0) = 0$ is possible. $C_g[0, \infty)$ is a subspace of $C[0, \infty)$. In these spaces we can consider the following cones

$$(2) \quad K[0, \infty) = \{x; x \in C[0, \infty), x(t) \geq 0 \text{ for } t \geq 0\}$$

and

$$(3) \quad K_g[0, \infty) = \left\{ x; x \in C_g[0, \infty), \frac{x(t)}{g(t)} \geq 0 \text{ for } t > 0 \right\}$$

with their corresponding interiors (in the topological sense)

$$(4) \quad \overset{\circ}{K}[0, \infty) = \{x; x \in C[0, \infty), m_0 \leq x(t) \leq M_0 \text{ for } t \geq 0\}$$

where $M_0 \geq m_0 > 0$ obviously depend of the function $x = x(t)$, and respectively

$$(5) \quad \overset{\circ}{K}_g[0, \infty) = \{x; x \in C_g[0, \infty), m_0 g(t) \leq x(t) \leq M_0 g(t) \text{ for } t > 0\}$$

where $M_0 \geq m_0 > 0$ again depend of the function $x = x(t)$. A slight generalization of Theorem 1.1 ([1], p. 330) asserts that if $T: K[0, \infty) \rightarrow K[0, \infty)$ is a monotone increasing mapping, homogeneous of degree p with $0 < p < 1$, for which there exists a continuous bounded function $g = g(t)$ such that $g(t) > 0$ for $t > 0$ and $Tg \in \overset{\circ}{K}_g[0, \infty)$, then T has a unique fixed point in $\overset{\circ}{K}_g[0, \infty)$. The proof is based upon the fact that with the pseudometric

$$(6) \quad d_g(x, y) = \log \frac{M(x/y)}{m(x/y)}, \quad x, y \in \dot{K}_g[0, \infty),$$

where

$$(7) \quad M(x/y) = \sup_{t>0} \frac{x(t)}{y(t)}, \quad m(x/y) = \inf_{t>0} \frac{x(t)}{y(t)},$$

the set $E_g[0, \infty)$ defined by

$$(8) \quad E_g[0, \infty) = \dot{K}_g[0, \infty) \cap \{x; x \in C_g[0, \infty), \|x\|_g = 1\}$$

becomes a complete metric space, in which the map $F(x) = Tx/\|Tx\|_g$ is a strict contraction. Now, we give two applications of the cited here result.

Theorem 1. Assume that

(i) $k = k(t, s) \geq 0$ is a continuous kernel for $0 \leq s \leq t < \infty$ having the property that for every $t_0 > 0$ there exists a number $s_0 \in (0, t_0)$ such that $k(t_0, s_0) > 0$;

(ii) there exists a constant $K > 0$, such that

$$(9) \quad \int_0^t k(t, s) \, ds \leq K \quad \text{for } t \geq 0;$$

(iii) there exists a continuous bounded function $g = g(t)$ on $[0, \infty)$, $g(0) = 0$ such that

$$(10) \quad mg(t) \leq \int_0^t k(t, s)(g(s))^p \, ds \leq Mg(t), \quad t \geq 0$$

where $0 < p < 1$ and $0 < m \leq M < \infty$.

Then the Volterra integral equation

$$(11) \quad x(t) = \int_0^t k(t, s)(x(s))^p \, ds, \quad t \geq 0$$

has a unique continuous solution $x = x(t)$ with the property

$$(12) \quad m_0 g(t) \leq x(t) \leq M_0 g(t), \quad t \geq 0,$$

where $0 < m_0 \leq M_0 < \infty$ are constants.

Proof. The map T given by

$$(13) \quad (Tx)(t) = \int_0^t k(t, s)(x(s))^p \, ds, \quad t \geq 0$$

satisfies all the required conditions in the cited above Theorem, if we take into account (i), (ii) and (iii). His unique fixed point $x \in \dot{K}_g[0, \infty)$ is the solution $x = x(t)$ of the equation (11) and is not difficult to observe that (12) is also fulfilled.

Remark 1. The equation (11) may be write in a equivalent form as follows

$$(14) \quad (y(t))^{1/p} = \int_0^t k(t, s)y(s)ds \quad t \geq 0 \quad 0 < p < 1$$

and consequently we can formulate a existence result about this last equation.

Another result is concerned with the Fredholm type integral equation

$$(15) \quad x(t) = \int_0^\infty k(t, s)(x(s))^p ds, \quad t \geq 0 \quad (0 < p < 1).$$

Theorem 2. Assume that

(i) $k = k(t, s) \geq 0$ is a continuous kernel for $t \geq 0, s \geq 0$ having the property that for every $t_0 > 0$ there exists $s_0 \geq 0$ such that $k(t_0, s_0) > 0$;

(ii) there exists a constant $K > 0$ such that

$$(16) \quad \int_0^\infty k(t, s)ds \leq K, \quad t \geq 0;$$

(iii) for every $t_0 \geq 0$ and $\varepsilon > 0$, there exists $A = A(\varepsilon, t_0)$ and $\delta = \delta(\varepsilon, t_0)$ such that

$$(17) \quad \int_A^\infty k(t_0 + \eta, s)ds < \varepsilon \quad \text{for} \quad |\eta| < \delta;$$

(iv) there exists a continuous bounded function $g = g(t)$ for $t \geq 0$, with $g(t) > 0$ for $t > 0$ such that

$$(18) \quad mg(t) \leq \int_0^\infty k(t, s)(g(s))^p ds \leq Mg(t), \quad t \geq 0,$$

where $0 < p < 1$ and $M \geq m > 0$ are constants. Then the equation (15) has a unique continuous solution $x = x(t)$ with the property

$$(19) \quad m_0 g(t) \leq x(t) \leq M_0 g(t), \quad t \geq 0,$$

where $0 < m_0 \leq M_0 < \infty$.

Proof. The map T , given by

$$(20) \quad (Tx)(t) = \int_0^\infty k(t, s)(x(s))^p ds, \quad t \geq 0,$$

acts from $K(0, \infty)$ and takes its values in $K[0, \infty)$. This assertion is implied by the conditions (i), (ii) and (iii) imposed to the kernel $k(t, s)$. The conditions (ii) and (iii) were used by C. A v r a m e s c u [3] to ensure the admis-

sibility of the pair $(C[0, \infty), C[0, \infty))$ with respect to the linear integral operator

$$(21) \quad (\mathcal{T}x)(t) = \int_0^{\infty} k(t, s)x(s)ds, \quad t \geq 0.$$

The remaining conditions (monotone increasing, homogeneous of degree p with $0 < p < 1$, $Tg \in \dot{K}_g[0, \infty)$) are also satisfied by the integral operator (20), whose unique fixed point in $\dot{K}_g[0, \infty)$ is the solution of the equations (15).

2. In this section, we shall construct a projective metric in some spaces of real sequences $x = (x_n)$. Denote by C the Banach space of all bounded sequences $x = (x_n)$ with the norm $\|x\| = \sup_{n \geq 1} |x_n|$ and by C_g the subspace of C consisting of all sequences $x = (x_n)$ for which $\|x\|_g = \sup_{n \geq 1} (|x_n|/g_n) < \infty$, where $g = (g_n)$ is a fixed bounded sequence with $g_n > 0$, $\forall n \in N =$ the set of natural numbers. We consider the cones $K = \{x; x = (x_n) \in C, x_n \geq 0 \text{ for every } n\}$ and $K_g = \{x; x = (x_n) \in C_g, x_n/g_n \geq 0 \text{ for every } n\}$, with their corresponding interiors $\dot{K} = \{x; x = (x_n) \in C, m_0 \leq x_n \leq M_0 \text{ for every } n\}$ where $M_0 > 0$ and $m_0 > 0$ depend of x and respectively $\dot{K}_g = \{x; x = (x_n) \in C_g, m_0 g_n \leq x_n \leq M_0 g_n \text{ for every } n\}$, where $M_0 > 0$ and $m_0 > 0$ again depend of the element x . For $x, y \in \dot{K}_g$, let

$$(22) \quad d_g(x, y) = \log \frac{M(x/y)}{m(x/y)},$$

where

$$(23) \quad M(x/y) = \sup_{n \geq 1} \frac{x_n}{y_n}, \quad m(x/y) = \inf_{n \geq 1} \frac{x_n}{y_n}$$

are positive numbers. The function d_g is a pseudometric in $\dot{K}_g$; $d_g(\lambda x, \mu y) = d_g(x, y)$ for all $\lambda > 0, \mu > 0$ and $d_g(x, y) = 0$ is equivalent with $x = \lambda y$ for some $\lambda > 0$.

Theorem 3. If we put $E_g = \dot{K}_g \cap \{x; x \in C_g, \|x\|_g = 1\}$, then

(i) $0 < m(x/y) \leq 1 \leq M(x/y) < \infty$;

(ii) $\frac{1}{2} m(y/g) \tanh \frac{1}{2} d_g(x, y) \leq \|x - y\|_g \leq \exp(d_g(x, y)) - 1$

for every $x, y \in E_g$.

Proof. This theorem is the analogue of the Lemma 3.1 ([1], p. 332) in the discrete case. We prove only the part (i) and the right inequality in (ii). As we remarked, $m(x/y)$ and $M(x/y)$ are positive numbers; it remains to show that $m(x/y) \leq 1 \leq M(x/y)$. If we suppose $m(x/y) > 1$, then

$$(24) \quad x_n \geq (1 + \delta)y_n, \quad \forall n \in N \quad \text{for some } \delta > 0$$

and therefore

$$(25) \quad 1 = \|x\|_g \geq (1 + \delta)\|y\|_g = 1 + \delta$$

which is a contradiction. This shows that $m(x/y) \leq 1$. Now, if we suppose $M(x/y) < 1$, then

$$(26) \quad x_n \leq (1 - \delta)y_n \quad \forall n \in N \quad \text{for some } \delta > 0,$$

which implies

$$(27) \quad 1 = \|x\|_g \leq (1 - \delta)\|y\|_g = 1 - \delta,$$

also impossible. This shows that $1 \leq M(x/y)$. Starting from the inequalities

$$(28) \quad m(x/y)y_n \leq x_n \leq M(x/y)y_n, \quad \forall n \in N,$$

we obtain

$$(29) \quad x_n - y_n \leq (M(x/y) - m(x/y))y_n,$$

and respectively

$$(30) \quad x_n - y_n \geq -(M(x/y) - m(x/y))y_n,$$

true for every n natural. The last two inequalities give

$$(31) \quad |x_n - y_n| \leq (M(x/y) - m(x/y))y_n, \quad \forall n \in N$$

and hence

$$(32) \quad \|x - y\|_g \leq M(x/y) - m(x/y) = (e^{d_g(x,y)} - 1)m(x/y) \leq e^{d_g(x,y)} - 1.$$

The first inequality in (ii) may be proved like as in ([1], p. 333). The proof is complete.

COROLLARY 1. *With the metric d_g , E_g is a complete metric space.*

THEOREM 4. *Assume that $T: K \rightarrow K$ is a monotone increasing mapping which is homogeneous of degree p , $0 < p < 1$. If there exists $g \in C$ with $g_n > 0$ for every n and $Tg \in \dot{K}_g$, then T has a unique fixed point in $\dot{K}_g$.*

PROOF. We show that $T: \dot{K}_g \rightarrow \dot{K}_g$. Indeed, if $x \in \dot{K}_g$ then for some positive constants m and M we have

$$(33) \quad mg_n \leq x_n \leq Mg_n, \quad \forall n \in N,$$

and because T is homogeneous of degree p , it follows

$$(34) \quad m^p(Tg)_n \leq (Tx)_n \leq M^p(Tg)_n, \quad \forall n \in N.$$

Taking into account that $Tg \in \dot{K}_g$, we can write

$$(35) \quad m_0 m^p g_n \leq (Tx)_{n_0} \leq M_0 M^p g_n, \quad \forall n \in N,$$

which shows that $Tx \in \dot{K}_g$. From the inequalities

$$(36) \quad m(x/y)y_n \leq x_n \leq M(x/y)y_n, \quad (m(x/y))^p(Ty)_n \leq (Tx)_n \leq (M(x/y))^p(Ty)_n$$

we conclude that $d_g(Tx, Ty) \leq p d_g(x, y)$ for every $x, y \in \dot{K}_g$. It follows that $F(x) = Tx / \|Tx\|_g$ is a strict contraction in $\dot{E}_g$. Consequently T has a unique fixed point in $\dot{K}_g$ and the proof is complete.

COROLLARY 2. *Assume that*

(i) $k_{ni} \geq 0$ for every natural n and i and there exists a constant $K > 0$ such that

$$(37) \quad \sum_{i=1}^{\infty} k_{ni} \leq K, \quad \forall n \in N;$$

(ii) there exists $g = (g_n) \in C$ with $g_n > 0$ for every n , such that

$$(38) \quad m g_n \leq \sum_{i=1}^{\infty} k_{ni} (g_i)^p \leq M g_n,$$

where $0 < p < 1$ and $m > 0$, $M > 0$. Then the Fredholm type discrete equation

$$(39) \quad x_n = \sum_{i=1}^{\infty} k_{ni} (x_i)^p, \quad n = \text{natural}$$

has a unique solution $x = (x_n)$ with the property

$$(40) \quad m_0 g_n \leq x_n \leq M_0 g_n, \quad \forall n \in N,$$

where $m_0 > 0$ and $M_0 > 0$ are constants.

Corollary 3. Assume that

(i) $k_{ni} \geq 0$ for every n and i natural;

(ii) there exists two positive constants m and M , such that

$$(41) \quad m \leq \sum_{i=1}^{\infty} k_{ni} \leq M, \quad \forall n \in N.$$

Then for every p with $0 < p < 1$, the Fredholm discrete equation (39) has a unique solution $x = (x_n)$ with the property $0 < m_0 \leq x_n \leq M_0 < \infty$.

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SOLUȚII POZITIVE PENTRU UNELE ECUAȚII FUNCȚIONALE

(Rezumat)

Folosind conceptul de metrică proiectivă se găsesc soluții în spații corespunzătoare de funcții sau de șiruri pentru ecuațiile (11), (15) și (39).

SUITES CONVERGENTES DÉFINIES PAR $x_{n+p} = f(x_n, x_{n+1}, \dots, x_{n+p-1})$ DANS DES ESPACES 2-MÉTRIQUES BORNÉS ET COMPLETS

PAR

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Définition 1. Un espace 2-métrique (X, d) est un ensemble X dans lequel pour chaque trois points arbitraires x, y, z il y a une fonction réelle $d: X \times X \times X \rightarrow R$ avec les propriétés :

(i) pour chaque paire de points $x, y \in X, x \neq y$ il y a $z \in X$ tel que $d(x, y, z) \neq 0$;

(ii) $d(x, y, z) = 0$ si et seulement si deux de ces trois points sont égaux ;

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$;

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z), \forall u \in X$.

Conséquence 1. d est une fonction nonnégative, c'est à dire $d: X \times X \times X \rightarrow R_+$. En effet, en posant $z = y$ dans (iv) on a

$$0 = d(x, y, y) \leq d(x, y, u) \leq d(x, u, y) + d(u, y, y).$$

De (ii), (iii) on obtient $0 = d(x, y, y) \leq d(x, y, u) + d(x, y, u)$. Donc $0 \leq 2d(x, y, u)$ ou $d(x, y, u) \geq 0$.

Définition 2. Si la fonction d est bornée, l'espace 2-métrique (X, d) s'appelle borné. Le diamètre de (X, d) , noté $\delta(X)$, est défini dans la manière suivante :

$$\delta(X) = \sup_{x, y, z \in X} d(x, y, z).$$

Définition 3. Si $d(x_n, x, a)$ est convergente à 0 pour tous $a \in X$, on dit que la suite (x_n) est convergente à x et x est une limite de (x_n) .

Conséquence 2. La limite d'une suite convergente dans un espace 2-métrique est unique.

En effet, supposons que (x_n) a deux limites $x, y \in X, x \neq y$, c'est à dire $d(x_n, x, a) \rightarrow 0, d(x_n, y, a) \rightarrow 0 (n \rightarrow \infty)$ pour tout $a \in X$. De l'axiome (iv) on a

$$(1) \quad d(x, y, a) \leq d(x_n, y, a) + d(x, x_n, a) + d(x, y, x_n).$$

Mais de la supposition précédente on a $d(x_n, x, y) \rightarrow 0$ pour $n \rightarrow \infty$ et tout $a \in X$. Donc, en vertu de (i) - (iii) on obtient $d(x, y, a) < \varepsilon$ pour tout $a \in X$, c'est à dire $d(x, y, a) = 0$.

De l'axiome (i) avec $x \neq y$ il suit qu'il y a $a \in X$ tel que $d(x, y, a) \neq 0$, ce qui contredit le résultat précédent, donc $x = y$.

Définition 4. Si dans un espace 2-métrique (X, d) on a $d(x_m, x_n, a) \rightarrow 0$ pour $m, n \rightarrow \infty$ pour tout $a \in X$, alors la suite (x_n) s'appelle *suite de Cauchy*. Si chaque suite de Cauchy est convergente, l'espace 2-métrique (X, d) s'appelle *complet*.

Maintenant nous introduisons la notion d'application p -orbital continue dans la manière suivante :

Définition 5. Soit $f: X^p \rightarrow X$, où (X, d) est un espace 2-métrique. Si $d(f(x_n, x_{n+1}, \dots, x_{n+p-1}), u, a) \rightarrow 0$ pour $n \rightarrow \infty$ et tout $a \in X$ implique $d(f(x_{n+1}, \dots, x_{n+p}), f(u, \dots, u), a) \rightarrow 0$ pour $n \rightarrow \infty$ et tout $a \in X$, alors la fonction f s'appelle *fonction p -orbital continue*.

Théorème 1. Si (X, d) est un espace 2-métrique borné et complet et $f: X^p \rightarrow X$ est une fonction p -orbital continue pour laquelle il y a p nombres réels $\alpha_1, \alpha_2, \dots, \alpha_p \geq 0$, $\sum_{i=1}^p \alpha_i = \alpha < 1$ tels qu'on a l'inégalité

$$(2) \quad d(f(u_1, u_2, \dots, u_p), f(u_2, u_3, \dots, u_{p+1}), a) \leq \alpha_1 d(u_1, u_2, a) + \alpha_2 d(u_2, u_3, a) + \dots + \alpha_p d(u_p, u_{p+1}, a)$$

pour tous $u_1, u_2, \dots, u_{p+1}, a \in X$, alors il existe un point unique $u \in X$ qui a la propriété

$$(3) \quad f(u, u, \dots, u) = u.$$

Démonstration. Considérons p éléments arbitraires de X et la suite (x_n) définie par la relation de récurrence

$$(4) \quad x_{n+p} = f(x_n, x_{n+1}, \dots, x_{n+p-1}) \quad (n = 1, 2, 3, \dots).$$

Posons $K = \max [d(x_1, x_2, a), \dots, d(x_p, x_{p+1}, a)]$. Évidemment $K \leq \delta(X)$. Alors

$$\begin{aligned} x_{p+1} &= f(x_1, x_2, \dots, x_p), \\ x_{p+2} &= f(x_2, x_3, \dots, x_{p+1}), \\ x_{2p} &= f(x_p, x_{p+1}, \dots, x_{2p-1}), \\ x_{2p+1} &= f(x_{p+1}, x_{p+2}, \dots, x_{2p}), \\ &\dots \end{aligned}$$

$$d(x_{p+1}, x_{p+2}, a) \leq \alpha_1 d(x_1, x_2, a) + \alpha_2 d(x_2, x_3, a) + \dots + \alpha_p d(x_p, x_{p+1}, a) \leq \alpha K,$$

$$d(x_{p+2}, x_{p+3}, a) \leq \alpha_1 d(x_2, x_3, a) + \dots + \alpha_p d(x_{p+1}, x_{p+2}, a) \leq$$

$$\leq \alpha \max [d(x_2, x_3, a), \dots, d(x_p, x_{p+1}, a)], \quad KI = \alpha K;$$

$$\dots \dots \dots d(x_{2p}, x_{2p+1}, a) \leq \alpha K.$$

En général nous obtenons

$$d(x_{pn+1}, x_{pn+2}, a) \leq \alpha^n K,$$

$$d(x_{pn+2}, x_{pn+3}, a) \leq \alpha^n K,$$

$$d(x_{p(n+1)}, x_{p(n+1)+1}, a) \leq \alpha^n K, \quad n = 1, 2, 3, \dots$$

Si $1 < m$ et $1 > p$, nous avons

$$d(x_1, x_m, a) \leq p \left\{ \alpha^{\left[\frac{1}{p}\right]} + \alpha^{\left[\frac{1}{p}\right]+1} + \dots \right\} K = \alpha^{\left[\frac{1}{p}\right]} \frac{pK}{1-\alpha},$$

pour tout $a \in X$.

Cette inégalité montre que la suite $(x_n) \subset X$, (X, d) étant espace 2-métrique borné et complet, est une suite de Cauchy. Donc la suite (x_n) est convergente à l'élément $u \in X$ et alors pour tout $a \in X$, $\lim_{n \rightarrow \infty} d(x_{n+p}, u, a) = 0$.

Mais f est une fonction p -orbital continue et donc de la définition 5 nous obtenons

$$\lim_{n \rightarrow \infty} d(f(x_{n+1}, \dots, x_{n+p}), f(u, u, \dots, u), a) = 0.$$

Pour démontrer que u satisfait à la relation (3), nous allons appliquer les axiomes (iii) et (iv) :

$$\begin{aligned} d(u, f(u, u, \dots, u), a) &\leq d(u, f(u, u, \dots, u), f(x_{n+1}, \dots, x_{n+p})) + d(u, f(x_{n+1}, \dots, x_{n+p}), a) + \\ &+ d(f(x_{n+1}, \dots, x_{n+p}), f(u, u, \dots, u), a) = d(x_{n+p+1}, u, f(u, u, \dots, u)) + \\ &+ d(x_{n+p+1}, u, a) + d(x_{n+p+1}, f(u, \dots, u), a) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

$$\text{pour } n \geq N_1, n_1 \geq N_2, n \geq N_3 \quad (N_1, N_2, N_3 \in \mathbb{N}),$$

car, pour tout $\varepsilon > 0$ et pour tout $a \in X$, il y a $N_1, N_2, N_3 \in \mathbb{N}$ tels que

$$d(x_{n+p+1}, u, f(u, u, \dots, u)) < \frac{\varepsilon}{3} \text{ pour } n \geq N_1, d(x_{n+p+1}, u, a) < \frac{\varepsilon}{3} \text{ pour } n \geq N_2$$

parce que $x_{n+p} \rightarrow u$ ($n \rightarrow \infty$) et $d(x_{n+p+1}, f(u, \dots, u), a) < \varepsilon/3$ pour $n \geq N_3$ parce que f est p -orbital continue. Donc

$$(5) \quad d(u, f(u, u, \dots, u), a) = 0 \text{ pour tout } a \in X.$$

Si $u \neq f(u, u, \dots, u)$, alors il existe, en vertu de l'axiome (i), $a \in X$ tel que $d(u, f(u, \dots, u), a) \neq 0$ ce qui contredit le résultat (5). Par conséquence $f(u, u, \dots, u) = u$.

Pour démontrer l'unicité du point $u \in X$ qui satisfait (5), nous supposons qu'il existe un autre point $v \in X$, $v \neq u$, tel que $f(v, v, \dots, v) = v$. Alors, en vertu de (ii) (iii) et (iv), on obtient

$$\begin{aligned} d(u, v, a) &= d(f(u, u, \dots, u), f(v, v, \dots, v), a) \leq \\ &\leq d(f(u, \dots, u), f(v, \dots, v), f(u, \dots, u, v)) + d(f(u, \dots, u), f(u, \dots, u, v), a) + \\ &+ d(f(u, \dots, u, v), f(v, \dots, v), a) \leq \\ &\leq \alpha_p d(u, v, v) + \alpha_p d(u, v, a) + d(f(u, \dots, u, v), f(v, \dots, v), a) = \\ &= \alpha_p d(u, v, a) + d(f(u, \dots, u, v), f(v, \dots, v), a) \leq \\ &\leq \alpha_p d(u, v, a) + \alpha_{p-1} d(u, v, a) + \dots + \alpha_1 d(u, v, a) = \alpha d(u, v, a). \end{aligned}$$

Donc $d(u, v, a) \leq \alpha d(u, v, a)$ où $\alpha < 1$ et $(1 - \alpha) d(u, v, a) \leq 0$. On sait que $1 - \alpha > 0$ et alors $d(u, v, a) \leq 0$. En vertu de la conséquence 1 il suit que

$$(6) \quad d(u, v, a) = 0.$$

Si $u \neq v$, il existe $a \in X$ tel que $d(u, v, a) \neq 0$ ce qui contredit (6). Donc $u = v$.

Observation 1. La notion de fonction orbital continue de [1] est 1-orbital continue dans le sens donné par nous (v. définition 5).

Observation 2. La condition (2) du Théorème 1 se réduit à $d(f(u_1), f(u_2), a) \leq \alpha d(u_1, u_2, a)$ dans le cas de $f: X \rightarrow X$, où (X, d) est un espace 2-métrique. Alors du Théorème 1 nous obtenons un théorème analogue du celui de contraction de Banach pour des espaces 2-métriques :

Théorème 2. Si (X, d) est un espace 2-métrique borné et complet et $f: X \rightarrow X$ est une fonction 1-orbital continue pour laquelle il existe α , $0 \leq \alpha < 1$ tel que

$$d(f(u_1), f(u_2), a) \leq \alpha d(u_1, u_2, a),$$

alors la fonction f a un point fixe unique $u \in X$. ($f(u) = u$).

Observation 3. La condition (2) du Théorème 1 implique la relation

$$(7) \quad d(f(u_1, \dots, u_p), f(u_2, \dots, u_{p+1}), a) \leq \alpha \max \{d(u_1, u_2, a), \dots, d(u_p, u_{p+1}, a)\}$$

pour tous $u_1, u_2, \dots, u_{p+1}, a \in X$, condition employée par nous dans la démonstration du Théorème 1.

Observation 4. La condition (2) est employée effectivement dans la démonstration de l'unicité du point $u \in X$ qui satisfait (3).

Observation 5. Dans cette Note, dans les conditions du Théorème 1 pour x et f , on a montré que toutes les suites définies en (4) par la relation

$$x_{n+p} = f(x_n, x_{n+1}, \dots, x_{n+p-1}),$$

où $(x_1, x_2, \dots, x_p)$ sont des points arbitraires dans X^p , sont convergentes au même point unique u qui satisfait à l'équation $f(u, u, \dots, u) = u$.

Cette observation justifie le titre de la Note.

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ȘIRURI CONVERGENTE DEFINITE PRIN $x_{n+p} = f(x_n, x_{n+1}, \dots, x_{n+p-1})$ ÎN SPAȚII 2-METRICE MĂRGINITE ȘI COMPLETE

(Rezumat)

Se stabilește rezultatul: dacă (X, d) este un spațiu 2-metric mărginit și complet și f este o funcție p -orbital continuă pentru care există p numere reale pozitive $\alpha_1, \alpha_2, \dots, \alpha_p$, cu $\sum_{i=1}^p \alpha_i = \alpha < 1$ astfel încât se verifică inegalitatea (2) pentru toți $u_1, u_2, \dots, u_p, u_{p+1}, a$ din X , atunci există un punct unic u în X , care satisface ecuația $f(u, u, \dots, u) = u$, la care converg toate șirurile definite prin $x_{n+p} = f(x_n, x_{n+1}, \dots, x_{n+p-1})$, unde $(x_1, x_2, \dots, x_p)$ sînt puncte arbitrare din X^p .

FIXED POINTS OF IMPLICIT CONTRACTIONS VIA CANTOR'S INTERSECTION THEOREM

BY

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1. Throughout this Note, we shall make use of the following set of general notations:

- i) $\mathfrak{A}(X) = \{Y \subseteq X; Y \neq \emptyset\}$, $\tilde{\mathfrak{A}}(X) = \{Y \in \mathfrak{A}(X); X \setminus Y \neq \emptyset\}$
- ii) $F(X, Y) = \{f; f: X \rightarrow Y\}$, $\varphi(T) = \{x \in X; x = Tx\}$,
 $\forall T \in F(X, X)$
- iii) $N = \{0, 1, \dots\}$, $N' = \{1, 2, \dots\}$, $\underbrace{X^n = X \times X \times \dots \times X}_n$, $\forall n \in N'$
- iv) $R =] - \infty, +\infty[$, $R_+ = [0, +\infty[$, $R_+^0 =]0, +\infty[$

On the other hand, if (X, d) is a metric space, denote

- v) $\bar{A}$ = the closure of A , $\forall A \in \mathfrak{A}(X)$
- vi) $d(x, A) = \inf\{d(x, y); y \in A\}$, $\forall x \in X$, $\forall A \in \mathfrak{A}(X)$
- vii) $\text{diam}(A) = \sup\{d(x, y); x, y \in A\}$, $\forall A \in \mathfrak{A}(X)$
- viii) $\mathfrak{M}(X) = \{A \in \mathfrak{A}(X); \text{diam}(A) < +\infty\}$

Remark 1.1. i) $d(x, A) = 0 \Leftrightarrow x \in \bar{A}$; ii) $\text{diam}(A) = \text{diam}(\bar{A})$.

The following fundamental result will be used in this Note

Theorem 1.1. (Cantor's intersection theorem). *Let (X, d) be a complete metric space and let $\{A(r); r > 0\} \subset \mathfrak{A}(X)$ be a family of sets which satisfies the following conditions*

$$(1.1) \quad \exists r_0 > 0 \text{ such that } \{A(r); 0 < r < r_0\} \cap \mathfrak{M}(X);$$

$$(1.2) \quad r_1, r_2 > 0, r_1 < r_2 \Rightarrow A(r_1) \subseteq A(r_2);$$

$$(1.3) \quad \lim_{r \rightarrow 0} [\text{diam } A(r)] = 0 \text{ (i.e., } \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon), \text{ such that, } r \in]0, \delta[\Rightarrow \text{diam}(A(r)) < \varepsilon).$$

Then, there exists a unique $z \in X$ such that $d(z, A(r)) = 0$, $\forall r > 0$.

Proof. It is sufficient to observe that the family $\{A(r); 0 < r < r_0\} \subseteq \mathfrak{M}(X)$ satisfies all the conditions of the classical Cantor's intersection theorem [2, p. 318]. Q.E.D.

2. In this paragraph, (X, d) is a given metric space.

Definition 2.1. A map $T \in F(X, X)$ is said to be a *globally asymptotic map* (abbreviated g.a.m.) on X iff $\inf \{d(x, Tx) ; x \in X\} = 0$.

Definition 2.2. A map $T \in F(X, X)$ is said to be a (Σ, g, P) -globally asymptotic map on X (where $\Sigma \subset F(X, X)$, $g \in F((K_+^0)^2, S)$, $P \in \tilde{\mathfrak{Q}}(S)$) iff, $\forall x \in X \setminus \varphi(T) \exists U \in \Sigma$ such that either $Ux \in \varphi(T)$, or $g(d(Ux, TUx), d(x, Tx)) \in P$.

Remark 2.1. If $T \notin \Sigma$ (respectively $\Sigma = \{T\}$) the term „globally“ is replaced with „spatially“ (respectively „temporally“); in other words, the family Σ (with $T \notin \Sigma$) defines the spatial directions in X , while the family $\{T\}$ defines the temporal direction in X .

Definition 2.3. A mapping $g \in F((K_+^0)^2, S)$ is said to be P -admissible (where $P \in \tilde{\mathfrak{Q}}(S)$) iff $\forall \{a_n ; n \in N\} \cap K_+^0$ with $g(a_{n+1}, a_n) \in P$, $\forall n \in N$, we have $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.1. Suppose that $g \in F((K_+^0)^2, S)$ and $P \in \tilde{\mathfrak{Q}}(S)$ satisfy

$$(2.1) \quad u, v \in K_+^0, \quad g(u, v) \in P \Rightarrow u \leq v;$$

$$(2.2) \quad \forall r > 0 \quad \exists a(r) \in]0, r[\text{ such that } u, v \in [r, r + a(r)[\Rightarrow g(u, v) \in S \setminus P.$$

Then, g is P -admissible.

Proof. Let $\{a_n ; n \in N\} \subset K_+^0$ be such that $g(a_{n+1}, a_n) \in P$, $\forall n \in N$. From (2.1), $a_1 \geq a_2 \geq \dots$, and therefore, $\lim_{n \rightarrow \infty} a_n = r$ exists. Suppose $r > 0$; then, there exists $n(r) \in N$ such that $\{a_n ; n \geq n(r)\} \subset [r, r + a(r)[$, which gives (from (2.2)) $g(a_{n+1}, a_n) \in S \setminus P$, $\forall n \geq n(r)$, contradicting the choice of this sequence. Q.E.D.

Theorem 2.1. Let $T \in F(X, X)$, $\Sigma \subset F(X, X)$, $g \in F((K_+^0)^2, S)$ and $P \in \tilde{\mathfrak{Q}}(S)$ be such that

$$(2.3) \quad T \text{ is a } (\Sigma, g, P)\text{-g.a.m. on } X;$$

$$(2.4) \quad g \text{ is } P\text{-admissible.}$$

Then, T is a g.a.m. on X .

Proof. Suppose that $\inf \{d(x, Tx) ; x \in X\} > 0$, and let $x \in X$ be given. By (2.3) we may construct a sequence $\{U_n ; n \in N\} \subset \Sigma$ and a sequence $\{x_n ; n \in N\} \subset X$, $x_0 = x$, $x_{n+1} = U_n(x_n)$ $\forall n \in N$, such that $g(a_{n+1}, a_n) \in P$, $\forall n \in N$, where $a_n = d(x_n, Tx_n)$, $\forall n \in N$. By (2.4) this implies $\lim_{n \rightarrow \infty} a_n = 0$, a contradiction. Q.E.D.

3. Definition 3.1. Let (X, d) be a metric space. A map $T \in F(X, X)$ is said to be a (f, P) -implicit contraction map (where $f \in F((K_+^0)^2 \times K_+^4, S)$ and $P \in \tilde{\mathfrak{Q}}(S)$) iff it satisfies

$$(3.1) \quad f(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), (dx, Ty), d(y, Tx)) \in P$$

for every $x, y \in X$ with $Tx \neq Ty$.

Definition 3.2. A function $f \in F((K_+^0)^2 \times K_+^4, S)$ is said to be (P, Q) -admissible (where $P \in \tilde{\mathfrak{Q}}(S)$ and $Q \in \tilde{\mathfrak{Q}}(K_+^4)$) iff it satisfies

A. Global conditions:

$$(3.2) \quad t, u > 0, v, w, p, q \geq 0 \Rightarrow f(t, u, v, p, q) = f(t, u, w, v, p, q) = f(t, u, v, w, q, p).$$

B. Local conditions : $\forall \varepsilon \in Q, \exists r = r(\varepsilon) \in]0, \varepsilon[$, such that

$$(3.3) \quad u > \varepsilon, t, p, q \in]u - r, u + r[, v, w \in]0, r[\Rightarrow f(t, u, v, w, p, q) \in S \setminus P,$$

$$(3.4) \quad v > \varepsilon, t, q \in]v - r, v + r[, u, w, p \in [0, r[\Rightarrow f(t, u, v, w, p, q) \in S \setminus P.$$

The main result of this Note is the following

Theorem 3.1. *Let (X, d) be a complete metric space and let $T \in F(X, X)$, $f \in F((R_+^0)^2 \times R_+^4, S)$, $P \in \mathfrak{Q}(S)$, $Q \in \mathfrak{Q}(R_+^0)$ be such that*

$$(3.5) \quad T \text{ is a g.a.m. on } X;$$

$$(3.6) \quad T \text{ is a } (f, P)\text{-i.c.m. on } X;$$

$$(3.7) \quad f \text{ is } (P, Q)\text{-admissible and } \inf Q = 0.$$

Then, T has a unique fixed point in X (or, in other words, $\varphi(T)$ is not empty and consists of a single element).

Proof. Suppose that $x \neq Tx, \forall x \in X$ and let us denote $A(r) = \{x \in X : d(x, Tx) < r\}, \forall r > 0$. By (3.5.), $A(r) \neq \emptyset, \forall r > 0$, and on the other hand, $0 < r_1 < r_2 \Rightarrow A(r_1) \subseteq A(r_2)$. Let $\varepsilon \in Q$ be fixed and let $r = r(\varepsilon)$ be given by Definition 3.2. We claim that $\text{diam}(A(r/2)) \leq \varepsilon$. Ideed, suppose that $\text{diam}(A(r/2)) > \varepsilon$; we may find $x, y \in A(r/2)$ such that $d(x, y) > \varepsilon$ (i.e., $d(x, Tx), d(y, Ty) \in]0, r/2[\subseteq]0, r[$, while $d(x, y) > \varepsilon$). This fact, combined with the triangle inequality yields

$$d(x, y) - r < d(x, y) - d(x, Tx) - d(y, Ty) \leq d(Tx, Ty) \leq d(x, y) + d(x, Tx) + d(y, Ty) < d(x, y) + r;$$

$$d(x, y) - r < d(x, y) - d(y, Ty) \leq d(x, Ty) \leq d(x, y) + d(y, Ty) < d(x, y) + r;$$

$$d(x, y) - r < d(x, y) - d(x, Tx) \leq d(y, Tx) \leq d(x, y) + d(x, Tx) < d(x, y) + r,$$

and this implies (from (3.3) + (3.1)) $S \setminus P \ni f(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \in P$, a contradiction, proving our assertion. Therefore, the family $\{A(r) : r > 0\} \subset P(x)$ satisfies all the conditions of Theorem 1.1 and so, there exists a unique $z \in X$ such that $d(z, A(r)) = 0, \forall r > 0$. Choose $\varepsilon \in]0, d(z, Tz)[\cap Q$ and let $r = r(\varepsilon)$ be taken from Definition 3.2. As $d(z, A(r/3)) = 0$, there exists $x \in A(r/3)$ with $d(z, x) < r/3$ (i.e., $d(z, x), d(x, Tx) \in]0, r/3[\subseteq]0, r[$). Now, from the evaluations

$$0 \leq d(z, Tx) \leq d(z, x) + d(x, Tx) < r,$$

$$d(z, Tz) - r < d(z, Tz) - d(z, x) - d(x, Tx) \leq d(Tz, Tx) \leq d(z, Tz) + d(z, x) + d(x, Tx) < d(z, Tz) + r,$$

$$d(z, Tz) - r < d(z, Tz) - d(z, x) \leq d(x, Tz) \leq d(z, Tz) + d(z, x) < d(z, Tz) + r$$

and from (3.4) + (3.1) we get $S \setminus P \ni f(d(Tz, Tx), d(z, x), d(z, Tz), d(x, Tx), d(z, Tx), d(x, Tz)) \in P$, a contradiction. This shows that the initial hypothesis (i.e., $x \neq Tx, \forall x \in X$) is false, i.e., T has at least a fixed point in X . The uniqueness of this fixed point follows from the evident equality $\{x \in X : x = Tx\} = \cap \{A(r) : r > 0\}$, and Theorem 1.1 quoted above. Q.E.D.

Remark 3.1. Conditions (3.5) may be replaced by

(3.5)' T satisfies conditions (2.3) + (2.4) of Theorem 2.1.

Remark 3.2. The main result of this Note might be compared with Theorem 1.1 of [3], and with a result of [1].

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PUNCTE FIXE DE CONTRACȚII IMPLICITE PRIN TEOREMA DE INTERSECȚIE A LUI CANTOR

(Rezumat)

Se demonstrează o teoremă de punct fix în condiții mai largi decât cele din teorema de intersecție a lui Cantor.

CIRCULAȚIE ÎN GRAFURI EVANTAI ÎNCHISE

DE

VIOREL GHIUȘ

Definiție. Numim graf evantai închis, un graf lob de forma

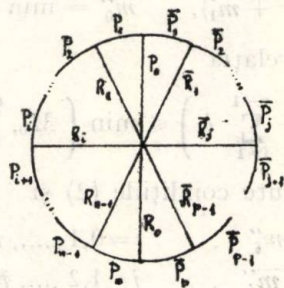


Fig. 1

unde P_i ($i = 0, 1, \dots, n$), $\bar{P}_j$ ($j = 1, 2, \dots, p$), R_i ($i = 1, 2, \dots, n-1$) $\bar{R}_j$ ($j = 1, 2, \dots, p-1$) sînt muchii sau drumuri simple.

Observație. Un graf evantai închis este un graf evantai [3] în care muchiile extremități coincid.

Vrem să studiem circulația în acest graf neorientat, știind că

$$(1) \quad \begin{cases} 0 \leq m_i \leq F(P_i) \leq M_i, & i = 0, 1, \dots, n, \\ 0 \leq \bar{m}_j \leq F(\bar{P}_j) \leq \bar{M}_j, & j = 1, 2, \dots, p, \\ 0 \leq l_i \leq F(R_i) \leq L_i, & i = 0, 1, \dots, n-1, \\ 0 \leq \bar{l}_j \leq F(\bar{R}_j) \leq \bar{L}_j, & j = 1, 2, \dots, p-1. \end{cases}$$

Definim cantitățile

$$l' = \max(l_0, m_n + \bar{m}_p), \quad l'' = \min(L_0, M_n + \bar{M}_p)$$

și considerăm relația

$$(2) \quad l' \leq l''.$$

Mai considerăm

$$(3') \begin{cases} m'_n = \begin{cases} m_n & \text{dacă } m_n + \bar{m}_p \geq l_0 \\ a & \text{dacă } m_n + \bar{m}_p < l_0 \end{cases} \\ \text{cu } a \geq m_n, \quad b \geq \bar{m}_p \text{ și } a + b = l_0 \\ m'_i = \max(m_i, m'_{i+1} + l_i), \\ i = 1, 2, \dots, n-1; \end{cases} \quad (3'') \begin{cases} \bar{m}'_p = \begin{cases} \bar{m}_p & \text{dacă } m_n + \bar{m}_p \geq l_0 \\ b & \text{dacă } m_n + \bar{m}_p < l_0 \end{cases} \\ \bar{m}_j = \max(\bar{m}_j, \bar{m}'_{j+1} + \bar{l}_j), \\ j = 1, 2, \dots, p-1; \end{cases}$$

$$(4') \begin{cases} m''_n = \begin{cases} M_n & \text{dacă } M_n + \bar{M}_p \leq L_0 \\ c & \text{dacă } M_n + \bar{M}_p > L_0 \end{cases} \\ \text{cu } c \geq m'_n, \quad d \geq \bar{m}'_p \text{ și } c + d = L_0 \\ m''_i = \min(M_i, m''_{i+1} + L_i), \\ i = 1, 2, \dots, n-1; \end{cases} \quad (4'') \begin{cases} \bar{m}''_p = \begin{cases} \bar{M}_p & \text{dacă } M_n + \bar{M}_p \leq L_0 \\ d & \text{dacă } M_n + \bar{M}_p > L_0 \end{cases} \\ \bar{m}''_j = \min(M_j, \bar{m}''_{j+1} + \bar{L}_j), \\ j = 1, 2, \dots, p-1, \end{cases}$$

Observație. Ca și în [3] sînt adevărate observațiile (1), (2) și (3). m'_n și $\bar{m}'_p$ pot fi definiți dacă este satisfăcută condiția (2);

$$(5) \quad m'_0 = \max(m_0, m'_1 + \bar{m}_1), \quad m''_0 = \min(M_0, m''_1 + \bar{m}''_1).$$

În sfîrșit mai considerăm relația

$$(6) \quad \max\left(m_0, \sum_{i=0}^{n-1} l_i + \sum_{j=1}^{p-1} l_j\right) \leq \min\left(M_0, \sum_{i=0}^{n-1} L_i + \sum_{j=1}^{p-1} \bar{L}_j\right).$$

În general nu sînt satisfăcute condițiile (2) și

$$(7) \quad \begin{cases} m'_i \leq m''_i, & i = 0, 1, \dots, n-1; \\ \bar{m}'_j \leq \bar{m}''_j, & j = 1, 2, \dots, p-1. \end{cases}$$

Teorema 1. Condiția necesară și suficientă ca într-un astfel de graf să existe o circulație de forma

$$(8) \quad \begin{cases} F(P_0) = F(P_1) + F(\bar{P}_1); \\ F(P_i) = F(P_{i+1}) + F(R_i), \quad i = 1, 2, \dots, n-1; \\ F(\bar{P}_j) = F(\bar{P}_{j+1}) + F(\bar{R}_j), \quad j = 1, 2, \dots, p-1; \\ F(R_0) = F(P_n) + F(\bar{P}_p) \end{cases}$$

satisfăcînd (1) este ca să fie îndeplinite condițiile (2), (6) și (7).

Necesitatea se demonstrează asemănător cu necesitatea din teorema 1 [3].

Suficiența. Definim $F(P_0) = s_0$ cu $m'_0 \leq s_0 \leq m''_0$, scriem $s_0 = s_1 + t_1$ cu $s_1 \geq m'_1$ și $t_1 \geq \bar{m}_1$ ceea ce este posibil deoarece $m'_0 \geq m'_1 + \bar{m}_1$ și punem $F(P_1) = s_1$ și $F(\bar{P}_1) = t_1$. Definim

$$F(P_{i+1}) = \max(m'_{i+1}, s_i - L_i) = s_{i+1}, \quad i = 1, 2, \dots, n-1;$$

$$F(R_i) = \begin{cases} s_i - m'_{i+1} & \text{dacă } m'_{i+1} \geq s_i - L_i, \\ L_i & \text{dacă } m'_{i+1} < s_i - L_i, \quad i = 1, 2, \dots, n-1; \end{cases}$$

$$F(\bar{P}_{j+1}) = \max(\bar{m}'_{j+1}, t_j - \bar{L}_j) = t_{j+1}, \quad j = 1, 2, \dots, p-1; \quad (VI)$$

$$F(\bar{R}_j) = \begin{cases} t_j - \bar{m}'_{j+1} & \text{dacă } \bar{m}'_{j+1} \geq t_j - \bar{L}_j, \\ \bar{L}_j & \text{dacă } \bar{m}'_{j+1} < t_j - \bar{L}_j, \end{cases} \quad j = 1, 2, \dots, p-1, \quad (VII)$$

iar $F(R_0) = F(P_n) + F(\bar{P}_p)$.

Se verifică ca în [3] pe baza lui (2), (6) și (7) că sînt satisfăcute (1) și (8).

Teorema 2. *Circulația în graful evantai închis din Teorema 1 poate fi redusă la o circulație într-un graf evantai de tipul celei din Teorema 2 [3] scindînd muchia R_0 în muchiile extremități R'_0 și $\bar{R}'_0$ în modul următor:*

dacă $l_0 \leq m_n + \bar{m}_p$, scriem $l'_0 = l'_0 + \bar{l}'_0$ cu $l'_0 \leq m_n = m'_n$ și $\bar{l}'_0 \leq \bar{m}_p = \bar{m}'_p$;

dacă $l_0 > m_n + \bar{m}_p$, scriem $l_0 = a + b$ cu $m'_n = a \geq m_n$ și $\bar{m}'_p = b \geq \bar{m}_p$;

dacă $L_0 \geq M_n + \bar{M}_p$, scriem $L_0 = L'_0 + \bar{L}'_0$ cu $L'_0 \geq M_n = m''_n$ și $\bar{L}'_0 \geq \bar{M}_p = \bar{m}''_p$;

dacă $L_0 < M_n + \bar{M}_p$, scriem $L_0 = c + d$ cu $m''_n = c \geq m'_n$ și $\bar{m}''_p = d \geq \bar{m}'_p$.

Se poate demonstra și reciproca Teoremei 2.

Teorema 3. *O circulație într-un graf evantai de tipul celei din Teorema 2 din [3] pentru $p = n$ poate fi redusă la o circulație într-un graf evantai închis de tipul celei din Teorema 1, făcînd muchiile extremități R'_0 și $\bar{R}'_0$ să coincidă în muchia R_0 și luînd*

$$l_0 = m_n + \bar{m}_p \quad \text{și} \quad L_0 = L_n + \bar{L}_p. \quad (VIII)$$

Fi: un graf evantai închis de forma

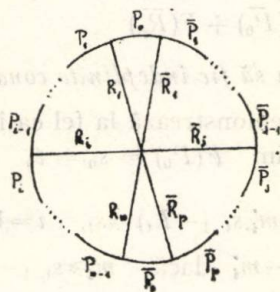


Fig. 2

Vrem să studiem circulația în acest graf neorientat știind că

$$(I) \quad \begin{cases} 0 \leq m_i \leq F(P_i) \leq M_i, & i = 0, 1, \dots, n-1; \\ 0 \leq \bar{m}_j \leq F(\bar{P}_j) \leq \bar{M}_j, & j = 0, 1, \dots, p-1; \\ 0 \leq l_i \leq F(R_i) \leq L_i, & i = 1, 2, \dots, n; \\ 0 \leq \bar{l}_j \leq F(\bar{R}_j) \leq \bar{L}_j, & j = 1, 2, \dots, p \end{cases} \quad \text{și}$$

$$(II) \quad \bar{m}_0 = \bar{M}_0.$$

Definim cantitățile

$$(III) \begin{cases} m'_n = \bar{m}_0 \\ m'_{i-1} = \max(m_{i-1}, m'_i + l_i), \end{cases} \quad \begin{cases} m''_n = \bar{M}_0 \\ m''_{i-1} = \min(M_{i-1}, m''_i + L_i), \end{cases} \quad i=2,3,\dots,n,$$

$$(IV) \begin{cases} \bar{m}'_p = \bar{m}_0 \\ \bar{m}'_{j-1} = \max(\bar{m}'_{j-1}, \bar{m}'_j + \bar{l}_j), \end{cases} \quad \begin{cases} \bar{m}''_p = \bar{M}_0 \\ \bar{m}''_{j-1} = \min(\bar{M}_{j-1}, \bar{m}''_j + \bar{L}_j), \end{cases} \quad j=2,3,\dots,p,$$

$$(V) \quad m'_0 = \max(m_0, m'_1 + l_1, \bar{m}'_1 + \bar{l}_1) \quad \text{și} \quad m''_0 = \min(M_0, m''_1 + L_1, \bar{m}''_1 + \bar{L}_1).$$

Mai considerăm relația

$$(VI) \quad \max \left(\sum_{i=1}^n l_i, \sum_{j=1}^p \bar{l}_j \right) \leq \left(\min \sum_{i=1}^n L_i, \sum_{j=1}^p \bar{L}_j \right).$$

În general nu sînt satisfăcute condițiile

$$(VII) \quad \begin{cases} m'_i \leq m''_i, & i=0,1,\dots,n-1, \\ \bar{m}'_j \leq \bar{m}''_j, & j=1,2,\dots,p-1. \end{cases}$$

Teorema 4. *Condiția necesară și suficientă ca într-un astfel de graf să existe o circulație de forma*

$$(VIII) \quad \begin{cases} F(P_1) + F(R_1) = F(P_0) = F(\bar{P}_1) + F(\bar{R}_1); \\ F(P_{i-1}) = F(P_i) + F(R_i) & i=2,3,\dots,n-1; \\ F(P_{n-1}) = F(\bar{P}_0) + F(R_n); \\ F(\bar{P}_{j-1}) = F(\bar{P}_j) + F(\bar{R}_j), & j=2,3,\dots,p-1; \\ F(\bar{P}_{p-1}) = F(\bar{P}_0) + F(\bar{R}_p) \end{cases}$$

satisfăcînd (I) și (II) este ca să fie îndeplinite condițiile (VI) și (VII).

Necesitatea se demonstrează la fel ca în Teorema 1 din [3].

Suficiența. Definim $F(P_0) = s_0 = t_0$ cu $m'_0 \leq s_0 = t_0 \leq m''_0$.

$$F(P_i) = \max(m'_i, s_{i-1} - L_i) = s_i, \quad i=1,2,\dots,n-1;$$

$$F(R_i) = \begin{cases} s_{i-1} - m'_i & \text{dacă } m'_i \geq s_{i-1} - L_i \\ L_i & \text{dacă } m'_i < s_{i-1} - L_i \end{cases} \quad i=1,2,\dots,n.$$

În particular

$$(IX') \quad F(\bar{P}_0) = \max(m'_n, s_{n-1} - L_n).$$

$$\text{Mai definim } F(\bar{P}_j) = \max(m'_n, t_{j-1} - \bar{L}_j), \quad j=1,2,\dots,p-1,$$

$$F(\bar{R}_j) = \begin{cases} t_{j-1} - \bar{m}'_j & \text{dacă } \bar{m}'_j \geq t_{j-1} - \bar{L}_j \\ \bar{L}_j & \text{dacă } \bar{m}'_j < t_{j-1} - \bar{L}_j \end{cases} \quad j=1,2,\dots,p.$$

În particular

$$(IX'') \quad F(\bar{P}_0) = \max(\bar{m}'_p, t_{p-1} - \bar{L}_p).$$

Din cauza lui (II) rezultă același $F(\bar{P}_0)$ din (IX') și (IX''). Se verifică ca în [3] pe baza lui (VI) și (VII) că sînt satisfăcute (I) și (VIII).

Teorema 5. *Circulația în graful evantai închis din Teorema 4 poate fi redusă la o circulație într-un graf evantai de tipul celei din Teorema 3 din [3] scindînd muchia $\bar{P}_0$ în muchiile extremităților P_n și $\bar{P}_p$ punînd $m_n = \bar{m}_0 = \bar{m}_p$ și $M_n = \bar{M}_0 = \bar{M}_p$.*

Se poate demonstra și reciproca Teoremei 5.

Teorema 6. *O circulație într-un graf de tipul celei din Teorema 3 din [3] cu $m_n = \bar{m}_p = M_n = \bar{M}_p$ poate fi redusă la o circulație într-un graf evantai închis de tipul celei din Teorema 4 făcînd muchiile extremități P_n și $\bar{P}_p$ să coincidă în muchia $\bar{P}_0$ și scriind $\bar{m}_0 = \bar{M}_0 = m_n$.*

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LA CIRCULATION DANS LES GRAPHS ÉVANTAILS FERMÉS

(Résumé)

On définit la notion de graphe évantail fermé et on étudie le problème de la circulation dans tel graphe. En utilisant ce fait on peut poser et résoudre le problème de la circulation dans des graphes évantails fermés à l'aide de la circulation dans des graphes évantails.

Ainsi on obtient des conditions nécessaires et suffisantes pour l'existence d'une circulation dans ces catégories de graphes.

REPREZENTAREA UMBRELOR ÎN SISTEMUL DE REFERINȚĂ PLAN AXIAL

DE

I. RUSU, CARMEN MARICA, MARIANA VASILACHE și CORNELIA EFTIMESCU

0. Introducere. În sistemul de referință plan-axial, sistem format dintr-un plan orizontal de cotă zero și o dreaptă perpendiculară pe acesta, un punct este reprezentat prin proiecțiile sale $A(a, \bar{a})$, care pot fi găsite cu ajutorul coordonatelor

carteziene $A\{a(x_A, y_A); \bar{a}(z_A)\}$ sau cilindrice: $A\{a(\rho_A, \theta_A); \bar{a}(z_A)\}$.

O dreaptă este determinată prin două puncte neconfundate ce-i aparțin, de exemplu $(D) \equiv (AB)$, în care $A(a, \bar{a})$ și $B(b, \bar{b})$, sînt cele două puncte, sau prin proiecția d și dreapta rabătuță pe planul (H) al reprezentării adică $D(d, D_0)$. Construcția în rabatură a dreptei, respectiv a poziției D_0 , presupune cunoașterea unui punct A de pe dreaptă, a unghiului θ_D pe care planul proiectant al dreptei îl face cu planul xOz și a unghiului φ_D dintre aceeași dreaptă și planul reprezentării. Prin urmare $D(d, D_0) = D(A, \theta_D, \varphi_D)$.

1. Umbra punctului. În sistemul de referință plan-axial s-a considerat planul $P(P_0, P_z)$, reprezentat prin urmele sale (fig. 1) și un punct $A(a, \bar{a})$, căruia urmează să i se găsească umbrele pe planele $[H]$ și $[P]$.

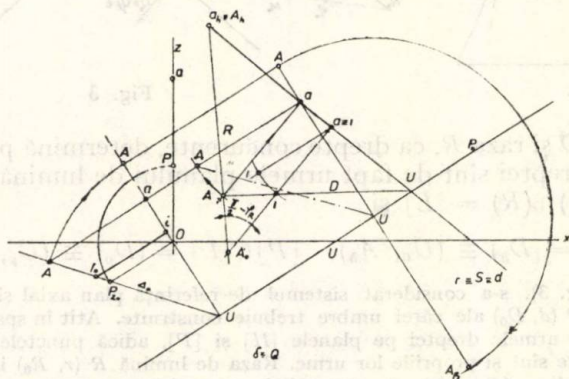


Fig. 1

În acest scop se consideră raza de lumină $R(r, R_0)$ care trece prin punctul A . Intersecția acestei drepte cu planele $[P]$ și $[H]$ sînt umbrele căutate ale punctului A . Prin urmare pe planul reprezentării $A_h = (R) \cap [H]$, iar pe planul $[P]$ de poziție generală $A_p = (R) \cap [P]$.

În epură s-a considerat prin $R(r, R_0)$ un plan proiectant vertical $[S]$, ce se intersectează cu planul $[P]$ după dreapta D . Această dreaptă este determinată de punctele $U_0 = (r) \cap (P_0)$ și $I(i, I_0)$, în care $iI_0 = z_I$, puncte de aparțin aceleiași orizontale a planului $[P]$. Punctul A rabătut pe planul $[H]$ are poziția A_0 , prin care se duce sub un unghi de $\pi/2 - \varphi_R$ în raport cu proiectanta (A_0a) , dreapta R_0 (raza de lumină rabătută în planul $[H]$).

În acest caz $A_p = (D_0) \cap (R_0)$, punct care este umbra punctului A pe planul $[P]$, dar rabătută pe planul $[H]$ al reprezentării. Proiecția acestui punct umbră pe planul $[H]$ este $a_p = (r) \cap (a_p A_p)$ în care $(A_p a_p) \perp (r)$ și este proiectanta lui A_p , iar $A_p a_p = z_{AP} = \text{cota}$ punctului umbră. Cunoscînd proiecția a_p a umbrei A_p se rabate acest punct în jurul axei P_0 , obținînd pozițiile A_{p1} și A_{p2} pe care le are umbra A_p după efectuarea rabaterii sale pe planul reprezentării $[H]$, ceea ce denotă că umbra A_p a punctului A a fost reprezentată în epură.

2. Umbra dreptei. În general fiecărui punct de pe o dreaptă îi corespunde o rază de lumină. Dacă sursa de lumină este infinită (improprie) atunci razele de lumină ale punctelor obiectului (dreptei) sînt paralele între ele și evident, dacă sursa de lumină are o poziție finită (sursă proprie) atunci razele de lumină corespunzătoare punctelor spațiului sînt concurente în punctul ce reprezintă sursa de lumină. Este suficient să se considere un singur punct al dreptei D , de exemplu punctul A (fig. 2), prin care se duce raza de lumină R .

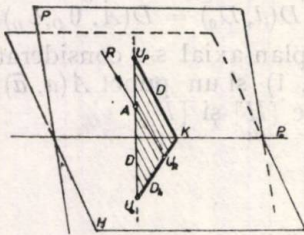


Fig. 2

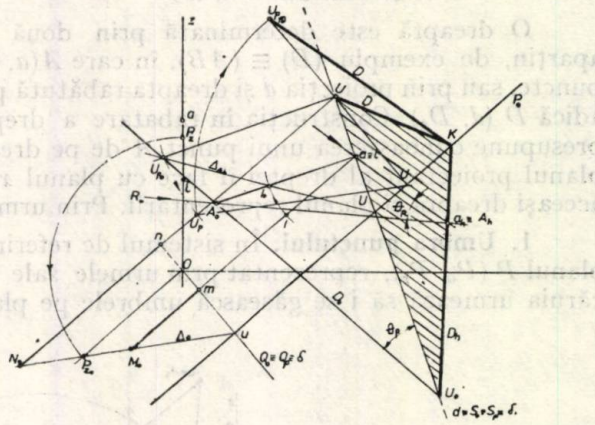


Fig. 3

Dreapta D și raza R , ca drepte concurente, determină planul de lumină $[L]$. Umbrele dreptei sînt de fapt urmele planului de lumină pe planele $[H]$ și $[P]$, deci $(D) \cap (R) = [L]$ și

$$[H] \cap [L] = [D_h] \equiv (U_0, A_h), \quad [P] \cap [L] = (D_p) \equiv (U_p, K).$$

În epură, (fig. 3), s-a considerat sistemul de referință plan axial și planul $P(P_0 P_z)$ precum și dreapta $D(d, D_0)$ ale cărei umbre trebuie construite. Atît în spațiu cît și în epură s-au găsit mai întîi urmele dreptei pe planele $[H]$ și $[P]$, adică punctele $U_0 = (D) \cap [H]$ și $U_p = (D) \cap [P]$ care sînt și propriile lor urme. Raza de lumină $R(r, R_0)$ intersectează planul $[H]$ în A_h . Practic $A_h = (R_0) \cap (r)$, R_0 construindu-se ca dreaptă ce trece prin A_0 și care face unghiul θ_R cu planul $[H]$, deci cu proiecția r . Urma U_0 s-a găsit la fel ca și urma A_h , adică

$U_{\bullet} = (d) \cap (D_0)$. Dreapta rabătuată D_0 s-a construit de asemenea prin punctul A , cu ajutorul unghiului θ_p . Pentru construcția urmei U_p , s-a utilizat procedeul specific de intersecție a unei drepte cu un plan.

În acest scop s-a dus prin (D) planul proiectant (vertical) $S(S_0, S_{z\infty})$, care se intersectează cu planul $[P]$ după dreapta $\Delta_1 (\delta_1, \Delta_{10})$. Intersecția $U_{p0} = (\Delta_{10}) \cap (D_0)$ este urma căutată, dar ea s-a găsit în poziția rabătuată pe planul reprezentării $[H]$. Dreapta de intersecție $\Delta_1 (\delta_1, \Delta_{10})$ s-a construit astfel: dacă $(\Delta_1) = [S] \cap [P]$ atunci $(\Delta_1) \in [P]$, urma sa pe planul $[H]$, deci un punct al ei este $U_{\delta} = (S_0) \cap (P_0)$. Al doilea punct este $I (i, I_0)$ a cărui cotă se poate afla, el fiind situat pe aceeași orizontală a planului ca și punctul $M (m, M_0)$ și deci $iI_0 = mM_0$ sau $z_I = z_M$ sau încă $(\Delta_p) \equiv (U_{\delta}, I_0)$.

Din punctul U_{p0} se duce perpendiculara pe d ; piciorul său U_p este proiecția pe planul $[H]$ a urmei drepte D pe planul $[P]$. Se observă că $(D_h) \equiv (U_0, A_h)$ și $(D_h) \cap (P_0) = K$ și în sfârșit $(D_p) \equiv (U_p, K)$. Pentru găsirea mărimii reale a umbrei U_{pK} din planul $[P]$ se rabate în planul $[H]$ această urmă astfel că $(D_{p0}) \equiv (U_{p10}, K)$, care este umbra reală căutată a dreptei D din planul $[P]$.

3. Umbra figurilor plane. 3.1. Umbra unui dreptunghi. Se consideră dreptunghiul $ABCE$ ($abce$, $\overline{abce}$) situat în planul de nivel $[H_1]$ și planul $P(P_0, P_z)$ în sistemul de referință plan axial.

Acest poligon plan are umbrele sale reale și virtuale pe planele $[H]$ și $[P]$ ca niște proiecții cilindrice pe aceste plane, în care scop se consideră umbra $A_h = (r) \cap R_0$ (fig. 5).

Se construiește dreptunghiul umbră $A_h B_h C_h E_h$ ca figură egală și de laturi corespunzătoare paralele cu dreptunghiul $ABCE$ dat.

Urma P_0 împarte în două zone umbra construită astfel: partea $B_h C_h K_1 K$ este umbra din fața planului $[P]$ iar partea $A_h E_h K_1 K$ este situată în spatele planului $[P]$, prima umbră fiind reală și cea de a doua virtuală. Prin urmare $(K K_1) \equiv (P_0)$ este linia de fringere.

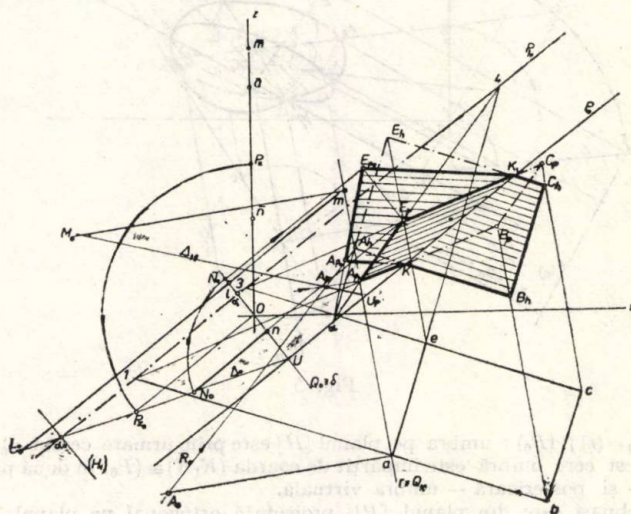


Fig. 4

Se intersectează apoi aceeași rază de lumină ce trece prin punctul $A (a, \bar{a})$ cu planul $[P_0]$. Se duce planul vertical $[Q_1]$ prin R . Atunci $(\Delta_1) = [R] \cap [P]$, dreaptă care în rabatere este Δ_1 .

Se consideră punctul $A_{p0} = (R_0) \cap (\Delta_{10})$, care este umbra punctului A pe planul $[P]$, rabătuată în planul $[H]$ al reprezentării. Prin ridicare acest punct ajunge în poziția A_{p0} , operație ce se realizează ducând prin A_{p0} perpendiculara pe $r \equiv Q_{10}$, A_p fiind piciorul acesteia pe urma și

proiecția menționată. Se construiește apoi axa de afinitate ca dreaptă comună a planelor $[P]$ și $[H_1]$, astfel: $(P_{H1}) = [P] \cap [H_1]$. Paralelogramul umbră $A_p B_p C_p E_p$ este figura afină a dreptunghiului $ABCE$, care este și umbra pe planul $[P]$ a acestui dreptunghi. Urma (axa de afinitate) P_h s-a construit prin punctul $(\alpha_1 \alpha_0)$, $\alpha_0 = [H_1] \cap (\Delta_0)$ și $\alpha \alpha_0 = a A_0 = z_1$. De asemenea dreapta $(KK_1) \equiv (P_0)$ este linia de frângere și deci de separare a umbrei reale de cea virtuală.

Umbra proiecție $A_p E_p K_1 K$ reală din planul $[P]$ se rabate apoi pe planul $[H]$ al reprezentării, folosind corespondența afină ce există între proiecția pe planul $[H]$ a dreptunghiului dat, proiecția umbrei sale pe planul $[P]$ și umbra reală din planul $[P]$ răbătută pe planul $[H]$, fiind necesară numai rabaterea proiecției E_p în poziția E_{p0} . S-a folosit astfel orizontala acestui punct și faptul că $z_E = z_N$ sau $z_E = n N_0$.

3.2. Umbra unui cerc. Cercul dat $\Omega (\omega, \Omega_0)$ este situat într-un plan de nivel $[H_1]$ de cotă $z_{H1} = \omega \Omega_0$. S-a reprezentat planul $P(P_0, P_z)$ umbrele urmînd a fi construite atît pe planul $[H]$ cît și pe planul $[P]$. În acest scop se duce raza de lumină $R (r, R_0)$ prin centrul $\Omega (\omega, \Omega_0)$, care intersectează planul reprezentării $[H]$ în ω_0 .

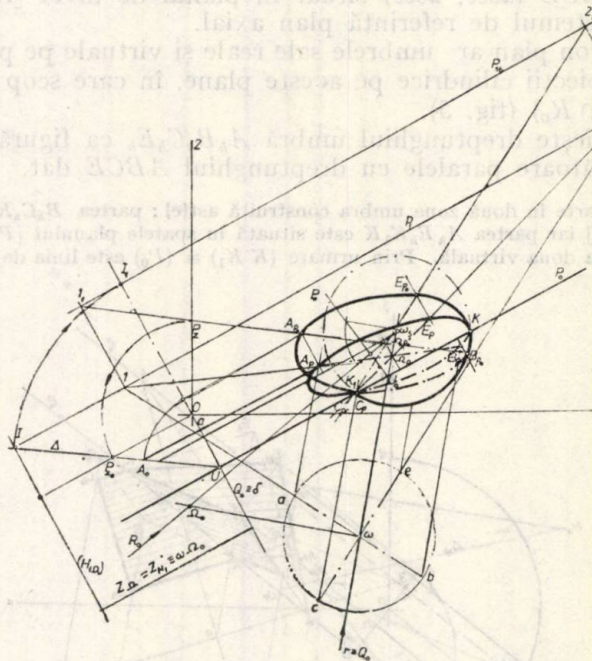


Fig. 5

În epură $\omega_0 = (r) \cap (R_0)$; umbra pe planul $[H]$ este prin urmare cercul ω_0 de aceeași rază cu cercul dat. Acest cerc umbră este împărțit de coarda $(K_1 K) \equiv (P_0)$ în două părți, anterioară — umbra reală — și posterioară — umbra virtuală.

Umbra aceluiași cerc din planul $[P]$, proiectată ortogonal pe planul $[H]$, se găsește intersectînd raza de lumină R cu acest plan. Construcția umbrei Ω_p se face ținînd seama că ea este o elipsă, curbă afină cercului Ω .

Se duce apoi urma $(H\Omega) \parallel (Q_0) \equiv \delta$, care intersectează dreapta Δ_0 în I , Δ_0 fiind linia cu cea mai mare pantă a planului $[P]$ ce trece prin urma P_z . Ducînd $(p_1) \parallel (P_0)$ se obține proiecția axei de afinitate. Se observă că (p_1) este intersectată de diametrii principali $AB(ab)$ și $CE(cc)$ în punctele 1 și 2 care se unesc prin linii drepte cu Ω_p , linii care se intersectează cu razele de lumină duse paralel cu (r) , obținînd diametrii umbră $A_p B_p$ și $C_p E_p$. Extremitățile lor împreună cu punctele K_1 și K permit trasarea elipsei umbră căutată.

Rabătind I în I_1 , în jurul urmei U , se poate duce urma $(P_1) = [P] \cap [H_1]$, astfel că $(P_1) \parallel (P_0)$. Perpendicularele din 1 și 2 la (p_1) și deci la (P_1) , sînt urmele planelor în care se află traiectoriile circulare ale rabaterii acestor puncte în jurul urmei P_0 . Unul din punctele elipsei Ω_p este și el rabătut odată cu planul $[P]$, de exemplu punctul A_p ajunge în A_{p0} . Dreapta care unește punctul I_1 cu punctul A_{p0} , este intersectată de perpendiculara la (P_0) dusă prin B_p , în punctul B_{p0} . Evident că $(B_p B_{p0})$ are aceeași semnificație ca și (11_1) ; (22_1) și $(\Omega_p \Omega_{p0})$. Dreapta $2_1 \Omega_{p0}$ este intersectată în punctele C_{p0} și E_{p0} de perpendicularele la (P_0) duse prin C_p și E_p .

Elipsa Ω_{p0} poate fi trasată căci i se cunosc diametrii conjugăți $A_{p0} B_{p0}$ și $C_{p0} E_{p0}$, precum și aceleași puncte ale liniei de fringere K_1 și K . În acest mod se găsește umbra pe planul $[P]$ a cercului Ω rabătut în planul $[H]$, deci în adevărată mărime.

4. Concluzii. 1. În sistemul de referință plan-axial, elementele și figurile geometrice plane sînt date prin proiecțiile lor pe planul reprezentării și pe axa reprezentării, utilizînd în acest scop rabaterea ca metodă uzuală pentru găsirea mărimii reale a unor unghiuri, distanțe și chiar a umbrei construite.

2. Pentru a cuprinde toată diversitatea de situații în care se cere construcția umbrei unui obiect considerat s-a introdus în reperul plan-axă și un plan P (P_c, P_z); umbrele construite au conturul real și virtual obținut prin secționarea cilindrului umbrei cu acest plan.

3. Lucrarea pune bazele teoriei umbrelor și aplicațiilor sale în sistemul de referință plan-axial, putînd fi utilă în proiectare și cercetare.

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REPRÉSENTATION DES OMBRES DANS LE SYSTÈME DE RÉFÉRENCE PLAN-AXIAL

(Résumé)

On présente la construction des ombres dans le système de référence plan-axial, où les éléments et les figures sont déterminées par leurs projections sur le plan et l'axe de la représentation. Pour couvrir toute la diversité des situations qui peuvent se présenter on introduit dans le système plan-axial un plan auxiliaire. L'intersection de ce plan avec le cylindre de l'ombre départage le contour réel de celui-ci, de son contour virtuel.

D.O. 539.31

DYNAMIC VIBRATION AND STRESSES IN NONHOMOGENEOUS CYLINDERS AND SPHERES

BY

J. G. CHAKRAVORTY

1. Introduction. Tranter [1] solved the dynamic thick elastic shell problems by means of Laplace transform while Baker and Allen [2] applied Mindlin-Goodman technique and solved the dynamic vibration problems for spherical shells under internal arbitrary loads. Cinelli [3] presented the theory of small deformation dynamic response of thick elastic shells to internally and externally applied loads which are arbitrary functions of space and time. In doing so he used the finite Hankel transform given by him [4] and thus avoided the complex inversion integrals associated with Laplace transform approach and the reduction of the boundary value problem into sub-problems associated with Mindlin-Goodman process.

The circular cylindrical and spherical shells are structures which find wide application in pressure vessels, buildings, aircrafts and missiles. For dynamic problems involving these structures, it is desirable to have solutions which apply equally to homogeneous and non-homogeneous shells. With this in view we solve in this paper the problems of

(i) radial motion of an infinitely long circular non-homogeneous cylindrical shell;

(ii) radially symmetrical motion of a non-homogeneous spherical shell.

The non-homogeneity of the isotropic thick shells are stipulated by the assumptions of the rigidity modulus μ and density ρ in the form

$$(1) \quad \mu = \mu_0 \left(\frac{r}{a} \right)^2 \quad \text{and} \quad \rho = \rho_0 \left(\frac{r}{a} \right)^2,$$

where μ_0 and ρ_0 are the values of μ and ρ respectively when $r = a$ (some assumed length).

In our approach we follow the method of Cenelli and observe that the determination of stresses and displacement may be made to depend upon the evaluation of elementary integrals in this case too which has been claimed by him to be an advantage of this method as applied to homogeneous medium.

2. Finite Hankel Transform. The formulae of the finite Hankel transform [4] are given below for later reference. A bar over lower case letters indicates the transform variable whereas a prime a lower case or capital letters denotes differentiation. H denotes the integral operator.

$$(A) \quad \bar{f}(\xi_i) = H[f(r)] = \int_a^b r f(r) C_m(r, \xi_i) dr, \quad a \leq r \leq b;$$

$$(B) \quad C_m(r, \xi_i) = \{J_m(\xi_i r) [\xi_i Y'_m(\xi_i a) + h Y_m(\xi_i a)] - Y_m(\xi_i r) [\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)],$$

$$(C) \quad f(r) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 [\xi_i J'_m(\xi_i b) + h J_m(\xi_i b)]^2 \bar{f}(\xi_i) \frac{C_m(r, \xi_i)}{F_m(\xi_i)},$$

$$(D) \quad \begin{aligned} \{F_m(\xi_i) = \{h^2 + \xi_i^2 \left[1 - \left(\frac{m}{\xi_i b}\right)^2\right]\} [\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]^2 - \\ - \{h^2 + \xi_i^2 \left[1 - \left(\frac{m}{\xi_i a}\right)^2\right]\} [\xi_i J'_m(\xi_i b) + h J_m(\xi_i b)]^2, \end{aligned}$$

where ξ_i is a positive root of

$$(E) \quad [\xi_i Y'_m(\xi_i a) + h Y_m(\xi_i a)] [\xi_i J'_m(\xi_i b) + h J_m(\xi_i b)] = [\xi_i Y'_m(\xi_i b) + h Y_m(\xi_i b)] [\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)],$$

$$(F) \quad H \left[\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{m^2}{r^2} f \right] = \frac{2 [\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{\pi [\xi_i J'_m(\xi_i b) + h J_m(\xi_i b)]} [f'(b) + h f(b)] - \frac{2}{\pi} [f'(a) + h f(a)] - \xi_i^2 \bar{f}(\xi_i),$$

where $J_m(\xi_i r)$, $Y_m(\xi_i r)$ are Bessel function of the first and second kind respectively of order m ; h , k are constant co-efficients whose values may be positive, negative or zero; a , b are inner and outer radii of the shell respectively; $f(r)$ is the arbitrary function of the variable r .

3. Radial Motion in Cylindrical Shell. The displacement components is cylindrical co-ordinates (r, θ, z) in the boundary value problem for dynamic radial vibration of a thick isotropic cylindrical shell of infinite length are

$$U_r = U(r, t), \quad U_\theta = U_z = 0.$$

The only non-vanishing equation of motion is then reduced to

$$(2) \quad \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \left[1 + \frac{r}{\mu} \frac{d\mu}{dr} \right] \frac{\partial U}{\partial r} - \left[1 - \frac{\nu r}{(1-\nu)\mu} \frac{d\mu}{dr} \right] \frac{U}{r^2} = \frac{(1-2\nu)}{2(1-\nu)} \frac{\rho}{\mu} \frac{\partial^2 U}{\partial t^2},$$

$$a \leq r \leq b, \quad t > 0,$$

in which μ and ρ are as given by (1) and ν is the constant Poisson's ratio.

The initial conditions are

$$(3) \quad U = \frac{\partial U}{\partial t} = 0, \quad t_i^1 = 0, \quad a \leq r \leq b,$$

and the boundary conditions are

$$(4) \quad \sigma_r(a, t) = \frac{2\mu}{1-2\nu} \left[(1-\nu) \frac{\partial U}{\partial r} + \nu \frac{U}{r} \right] = A(t), \quad r=a, \quad t > 0,$$

$$(5) \quad \sigma_r(b, t) = \frac{2\mu}{1-2\nu} \left[(1-\nu) \frac{\partial U}{\partial r} + \nu \frac{U}{r} \right] = B(t), \quad r=b, \quad t > 0,$$

where $A(t)$ and $B(t)$ are the radial stresses on the inner and outer surfaces respectively.

Using $U = r^{-1}V$ in (2) where V is a function of r and t , we have

$$(6) \quad \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{m^2}{r^2} V = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2},$$

in which $m^2 = 2(1-2\nu)/(1-\nu)$ and $c^2 = 2(1-\nu)\mu_0/(1-2\nu)\rho_0$, c being the velocity of irrotational waves in a medium with constant shear modulus μ_0 and constant density ρ_0 . It should be noted that

$$0 < m^2 < 2, \quad \text{for} \quad 0 < \nu < \frac{1}{2}.$$

The initial and boundary condition (3), (4) and (5) now become

$$(7) \quad V = \frac{\partial V}{\partial t} = 0, \quad t = 0, \quad a \leq r \leq b,$$

$$(8) \quad \frac{2\mu_0}{a^2(1-2\nu)} \left[(1-\nu)r \frac{\partial V}{\partial r} - (1-2\nu)V \right] = A(t), \quad r=a, \quad t > 0,$$

$$(9) \quad \frac{2\mu_0}{a^2(1-2\nu)} \left[(1-\nu)r \frac{\partial V}{\partial r} - (1-2\nu)V \right] = B(t), \quad r=b, \quad t > 0.$$

Using the definitions

$$h = \frac{-2\mu_0}{2\mu_0(1-\nu)a} \quad \text{and} \quad k = \frac{-2\mu_0}{2\mu_0(1-\nu)b}$$

we get from (F)

$$(10) \quad \begin{aligned} & H \left[\frac{d^2 V}{dr^2} + \frac{1}{r} \frac{dV}{dr} - \frac{m^2}{r^2} V \right] = \\ & = \frac{2}{\pi} \frac{(1-2\nu)}{2\mu_0(1-\nu)} \left\{ b^{-1} \frac{[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} \times \right. \\ & \quad \times \frac{2\mu_0}{a^2(1-2\nu)} [(1-\nu)bV'(b) - (1-2\nu)V(b)] - \\ & \quad \left. - a^{-1} \frac{2\mu_0}{a^2(1-2\nu)} [(1-\nu)aV'(a) - (1-2\nu)V(a)] \right\} - \xi_i^2 \bar{V}(\xi_i). \end{aligned}$$

Comparing (8), (9) and (10) the appropriate finite Hankel transform for (6) is

$$(11) \quad \bar{V} = \bar{V}(\xi_i, t) = \int_a^b r V(r, t) C_m(r, \xi_i) dr,$$

where C_m is given by (B) for $0 < m^2 < 2$.

Applying (11) to (6) and using (8), (9) and (10) we have

$$(12) \quad \frac{1}{c^2} \frac{\partial^2 \bar{V}}{\partial t^2} = \frac{1-2\nu}{\pi \mu_0 (1-\nu)} \left\{ b^{-1} \frac{[\xi_i J'_m(\xi_i a) + h(J_m(\xi_i a))]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} \sigma_r(b, t) - \right. \\ \left. - a^{-1} \sigma_r(a, t) \right\} - \xi_i^2 \bar{V}(\xi_i).$$

Rearranging (12) and putting the surface loads we get

$$(13) \quad \frac{1}{c^2} \frac{\partial^2 \bar{V}}{\partial t^2} + \xi_i^2 \bar{V}(\xi_i) = \\ = \frac{2}{\pi c \rho_0 c^3} \left\{ b^{-1} \frac{[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} B(t) - a^{-1} A(t) \right\}.$$

Now we use the Laplace transform, initial conditions and the convolution integral to get the solution of (13) as

$$(14) \quad \bar{V}(\xi_i, t) = \frac{2}{\pi c \rho_0 \xi_i} \int_0^t \left\{ \frac{[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} b^{-1} B(\tau) - \right. \\ \left. - a^{-1} A(\tau) \right\} \sin \xi_i c(t-\tau) d\tau.$$

From (C) we then have

$$V(r, t) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 [\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]^2 \bar{V}(\xi_i, t) \frac{C_m(r, \xi_i)}{F_m(\xi_i)},$$

where ξ_i is given by (E) for $0 < m^2 < 2$ and F_m is given by (D) for $0 < m^2 < 2$.

Thus the solution to the original problem is

$$(15) \quad U(r, t) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 [\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]^2 \frac{\bar{V}(\xi_i, t)}{F_m(\xi_i)} \frac{C_m(r, \xi_i)}{r}.$$

Placing finally (14) into (15) we get the solution for $\nu=1/3$ and $m=1$ as

$$\begin{aligned}
 U(r, t) &= \frac{\pi}{2\sqrt{\rho_0\mu_0}} \sum_{\xi_i} \xi_i [\xi_i J_1'(\xi_i b) + k J_1(\xi_i b)]^2 \frac{C_1(r, \xi_i)}{r F_1(\xi_i)} \times \\
 (16) \quad &\times \int_0^t \left\{ \frac{[\xi_i J_1'(\xi_i a) + h J_1(\xi_i a)]}{[\xi_i J_1'(\xi_i b) + k J_1(\xi_i b)]} b^{-1} B(\tau) - a^{-1} A(\tau) \right\} \sin 2\xi_i \sqrt{\frac{\mu_0}{\rho_0}} (t - \tau) d\tau.
 \end{aligned}$$

In this equation (A) to (E) are suitably changed for $m = 1$. Using (16) the stresses are found as

$$\begin{aligned}
 \sigma_r &= \pi c_0 S I \left\{ 2r^2 \frac{\partial}{\partial r} \left[\frac{C_1(\xi_i, r)}{r} \right] + C_1(\xi_i, r) \right\}, \\
 \sigma_\theta &= \pi c_0 S I \left\{ r^2 \frac{\partial}{\partial r} \left[\frac{C_1(\xi_i, r)}{r} \right] + 2C_1(\xi_i, r) \right\}, \\
 \sigma_z &= \pi c_0 S I \left\{ r^2 \frac{\partial}{\partial r} \left[\frac{C_1(\xi_i, r)}{r} \right] + C_1(\xi_i, r) \right\},
 \end{aligned}$$

where

$$S = \sum_{\xi_i} \xi_i \frac{[\xi_i J_1'(\xi_i b) + k J_1(\xi_i b)]^2}{F_1(\xi_i)},$$

$$I = \int_0^t \left\{ \frac{[\xi_i J_1'(\xi_i a) + h J_1(\xi_i a)]}{[\xi_i J_1'(\xi_i b) + k J_1(\xi_i b)]} b^{-1} B(\tau) - a^{-1} A(\tau) \right\} \sin 2\xi_i c_0 (t - \tau) d\tau,$$

and

$$c_0 = \sqrt{\frac{\mu_0}{\rho_0}}.$$

5. Radial Motion in Spherical Shell. For the radial vibration of a thick spherical shell we assume the displacement components in spherical polar co-ordinates (r, θ, Φ) as

$$U_r = v(r, t), \quad U_\theta = U_\Phi = 0.$$

The non-vanishing equation of motion in this case is

$$\begin{aligned}
 (17) \quad \frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \left[1 + \frac{r}{2\mu} \frac{d\mu}{dr} \right] \frac{\partial v}{\partial r} - \frac{2}{r^2} \left[1 - \frac{vr}{(1-\nu)\mu} \frac{d\mu}{dr} \right] v = \\
 = \frac{(1-2\nu)}{2(1-\nu)} \frac{\rho}{\mu} \frac{\partial^2 v}{\partial t^2}, \quad a \leq r \leq b,
 \end{aligned}$$

with the initial condition

$$(18) \quad v = \frac{\partial v}{\partial t} = 0, \quad t = 0, \quad a \leq r \leq b$$

and the boundary conditions

$$(19) \quad \sigma_r(a, t) = \frac{2\mu}{1-2\nu} \left[(1-\nu) \frac{\partial v}{\partial r} + 2\nu \frac{v}{r} \right] = A(t), \quad r = a, \quad t > 0;$$

$$(20) \quad \sigma_r(b, t) = \frac{2\mu}{1-2\nu} \left[(1-\nu) \frac{\partial v}{\partial r} + 2\nu \frac{v}{r} \right] = B(t), \quad r = b, \quad t > 0.$$

Using $v = r^{-\frac{3}{2}} \Psi(r, t)$ we get by (17) and (1)

$$(21) \quad \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} - \frac{m^2}{r^2} \Psi = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2},$$

in which $m^2 = (17 - 33\nu)/4(1 - \nu)$ and c has the same definition as before. We have

$$1 < 4m^2 < 17, \quad \text{for } 0 < \nu < \frac{1}{2}.$$

The initial and boundary conditions become

$$(22) \quad \Psi = \frac{\partial \Psi}{\partial t} = 0, \quad t = 0, \quad a \leq r \leq b,$$

$$(23) \quad \frac{\mu_0}{a^2(1-2\nu)} \left[2(1-\nu)r^{\frac{1}{2}} \frac{\partial \Psi}{\partial r} + (7\nu-3)r^{-\frac{1}{2}} \Psi \right] = A(t), \quad r = a, \quad t > 0,$$

$$(24) \quad \frac{\mu_0}{a^2(1-2\nu)} \left[2(1-\nu)r^{\frac{1}{2}} \frac{\partial \Psi}{\partial r} + (7\nu-3)r^{-\frac{1}{2}} \Psi \right] = B(t), \quad r = b, \quad t > 0.$$

Using the definitions

$$(25) \quad h = \frac{\mu_0(7\nu-3)/(1-2\nu)\sqrt{a}}{2\mu_0(1-\nu)\sqrt{a}/(1-2\nu)}, \quad k = \frac{\mu_0(7\nu-3)/(1-2\nu)\sqrt{b}}{2\mu_0(1-\nu)\sqrt{b}/(1-2\nu)}$$

we get from (F)

$$(26) \quad H \left[\frac{d^2 \Psi}{dr^2} + \frac{1}{r} \frac{d\Psi}{dr} - \frac{m^2}{r^2} \Psi \right] = \frac{1-2\nu}{\pi\mu_0(1-\nu)} \left\{ b^{-\frac{1}{2}} \frac{[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} \times \right. \\ \left. \times \frac{\mu_0}{a^2(1-2\nu)} [2(1-\nu)b^{\frac{1}{2}} \Psi'(b) + (7\nu-3)b^{-\frac{1}{2}} \Psi(b)] - \right. \\ \left. - a^{-\frac{1}{2}} \frac{\mu_0}{a^2(1-2\nu)} [2(1-\nu)a^{\frac{1}{2}} \Psi'(a) + (7\nu-3)a^{-\frac{1}{2}} \Psi'(a)] \right\} - \xi_i^2 \bar{\Psi}(\xi_i).$$

Comparing (23), (24) and (25) the appropriate finite Hankel transform for (21) is

$$(27) \quad \bar{\Psi} = \bar{\Psi}(\xi_i, t) = \int_a^b r \Psi(r, t) C_m(r, \xi_i) dr,$$

where C_m is given by (B) and $1 < 4m^2 < 17$.

Applying (27) to (24) and using (23) (24) and (26) we get

$$(28) \quad \frac{1}{c^2} \frac{\partial^2 \bar{\Psi}}{\partial t^2} + \xi_i^2 \bar{\Psi} = \frac{2}{\pi \rho_0 c^2} \left\{ b^{-\frac{1}{2}} \frac{[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} B(t) - a^{-\frac{1}{2}} A(t) \right\}.$$

The solution of this equation is obtained in the same manner as in the case of a cylindrical shell. Then by (C) we can easily find $\Psi(r, t)$. Then $v(r, t)$ is obtained as

$$(29) \quad v(r, t) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 [\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]^2 \frac{\bar{\Psi}(\xi_i, t) C_m(r, \xi_i)}{F_m(\xi_i) r^{3/2}},$$

where $\bar{\Psi}(\xi_i, t)$ being the solution of (28), is

$$(30) \quad \bar{\Psi}(\xi_i, t) = \frac{2}{\pi \rho_0 c \xi_i} \int_0^t \left\{ \frac{[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]} b^{-\frac{1}{2}} B(\tau) - a^{-\frac{1}{2}} A(\tau) \right\} \sin \xi_i c(t - \tau) d\tau.$$

Placing (30) in (29) we get finally the solution for $\nu = 1/3$ and $m = 3/2$ as

$$v(r, t) = \frac{\pi}{2\sqrt{\rho_0} c_0} \sum_{\xi_i} \xi_i \frac{[\xi_i J'_{3/2}(\xi_i b) + k J_{3/2}(\xi_i b)]^2}{F_{3/2}(\xi_i)} \frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \times \\ \times \int_0^t \left\{ \frac{[\xi_i J'_{3/2}(\xi_i a) + h J_{3/2}(\xi_i a)]}{[\xi_i J'_{3/2}(\xi_i b) + k J_{3/2}(\xi_i b)]} b^{-\frac{1}{2}} B(\tau) - a^{-\frac{1}{2}} A(\tau) \right\} \sin 2\xi_i c_0(t - \tau) d\tau.$$

The stresses are then given as

$$\sigma_r = 2\pi c_0 S_1 I_1 \left\{ r^2 \frac{\partial}{\partial r} \left[\frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \right] + \frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \right\}, \\ \sigma_\theta = 2\pi c_0 S_1 I_1 \left\{ r^2 \frac{\partial}{\partial r} \left[\frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \right] + \frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \right\}, \\ \sigma_\Phi = \pi c_0 S_1 I_1 \left\{ r^2 \frac{\partial}{\partial r} \left[\frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \right] + \frac{C_{3/2}(r, \xi_i)}{r^{3/2}} \right\},$$

where

$$S_1 = \sum_{\xi_i} \xi_i \frac{[\xi_i J'_{3/2}(\xi_i b) + k J_{3/2}(\xi_i b)]^2}{F_{3/2}(\xi_i)}, \\ I_1 = \int_0^t \left\{ \frac{[\xi_i J'_{3/2}(\xi_i a) + h J_{3/2}(\xi_i a)]}{[\xi_i J'_{3/2}(\xi_i b) + k J_{3/2}(\xi_i b)]} b^{-\frac{1}{2}} B(\tau) - a^{-\frac{1}{2}} A(\tau) \right\} \sin 2\xi_i c_0(t - \tau) d\tau.$$

It is interesting to note that $\sigma_r = 2\sigma_\theta = 2\sigma_\Phi$.

6. **Conclusion.** Thus we see that all the integrals to be evaluated to find the stresses and displacement are of the form

$$\int_0^t F(\tau) \sin \xi_0 c_0 (t - \tau) d\tau.$$

For specific $F(\tau)$ the integrals are elementary and can be evaluated easily. Then the stresses and displacements can be found out.

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VIBRAȚII ȘI TENSIUNI ÎN CILINDRI ȘI SFERE NEOMOGENE

(Rezumat)

Se utilizează transformata Hankel finită pentru a obține deplasările și tensiunile care se dezvoltă în plăcile sferice și cilindrice subțiri, sub acțiunea unor sarcini dinamice, funcții arbitrare de timp, aplicate pe suprafață.

PULSATILE FLOW OF A MICROPOLAR FLUID IN TUBES OF VARYING CROSS-SECTIONS

BY

VALERIU A. SAVA

1. Introduction. The classical fluid mechanic problem in the study of blood circulation is that of a pulsatile flow in an elastic tube. In studying blood flow in arteries, Womersley [1] studied oscillating flow in a thin walled elastic tube. Detailed measurements of the oscillating velocity profiles were made by Lindford and Ryan [2]. In all these analyses the flow is considered in cylindrical pipes of uniform cross-section.

It is well known that the blood vessels bifurcate at frequent intervals and the diameter of the vessels vary with distance. Thus the concept of flow in a slowly varying cross-section forms the basis of a large class of problems in bio-mechanics. In recent years this concept has been applied to study the flows in a locally constricted tube by Lee and Fung [3], in a peristaltic tube by Burns and Parkes [4] and in a slowly varying tube by Manton [5], Ramachandra Rao and Devanathan [6].

All these papers have treated blood as a homogeneous, incompressible viscous liquid governed by Navier-Stokes equations. In the last years a number of authors [7], [8], [9] studied blood flow based on the micropolar theory as been formulated by Eringen [10].

In the present work an attempt is made to understand the blood flow in a vascular bed by studying the fluid mechanical aspects of the pulsatile flow of a micropolar fluid in tubes of varying cross-section. The solutions of governing partial differential equations are evaluated by taking the amplitude of the oscillations to be very small for tubes of arbitrary cross sections, which vary slowly in the axial direction. Since the determination of the shear stress distribution of the wall of a arteriosclerotic blood vessel is of practical interest, we have obtained the expression for the shearing stress on the wall of the tube.

2. Problem Formulation. The flow of a micropolar fluid is governed by the following field of equations due in the vectorial form [10]

$$(1) \quad (\lambda + \mu) \nabla \nabla \cdot \mathbf{v} + (\mu + k) \nabla^2 \mathbf{v} + k \nabla \times \mathbf{v} - \nabla p + \rho \mathbf{f} = \rho \dot{\mathbf{v}},$$

$$(2) \quad (\alpha + \beta) \nabla \nabla \cdot \mathbf{v} + \gamma \nabla^2 \mathbf{v} + k \nabla \times \mathbf{v} - 2k \mathbf{v} + \rho \mathbf{l} = \rho j \dot{\mathbf{v}},$$

$$(3) \quad \dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0,$$

where $\mathbf{v}$ is the velocity, $\mathbf{v}$ the micro-rotation or spin, p the thermodynamic pressure, $\mathbf{f}$ and $\mathbf{l}$ the body-force and -couple per unit mass, ρ the density and j the micro-inertia; $\lambda, \mu, k, \alpha, \beta$ and γ are the material constants, or viscosity coefficients; the dot signifies material differentiation.

The pulsatile motion of an incompressible micropolar fluid in an infinite axisymmetric tube of varying cross-section is considered. We choose cylindrical polar coordinates (r, θ, z) such that $r = 0$ is the axis of symmetry for the tube and $r = a(z)$, an arbitrary function of z , is the radius of the cross-section of the tube. We seek solutions to equations (1), (2), (3) of the form

$$(4) \quad \begin{aligned} v_r &= v(r, z, t), & v_\theta &= 0, & v_z &= w(r, z, t), \\ v_r &= 0, & v_\theta &= \nu(r, z, t), & v_z &= 0. \end{aligned}$$

Substituting (4) into (1)–(3), we obtain the following coupled partial differential equations for $v(r, z, t)$, $w(r, z, t)$ and $\nu(r, z, t)$

$$(5) \quad \begin{aligned} -(\mu + k) \frac{1}{r} \frac{\partial(r\omega)}{\partial z} + \rho w \omega + k \frac{1}{r} \frac{\partial(r\nu)}{\partial z} &= -\frac{\partial H}{\partial r} - \rho \frac{\partial v}{\partial t}, \\ (\mu + k) \frac{1}{r} \frac{\partial(r\omega)}{\partial r} - \rho v \omega - k \frac{1}{r} \frac{\partial(r\nu)}{\partial r} &= -\frac{\partial H}{\partial z} - \rho \frac{\partial w}{\partial t}, \end{aligned}$$

$$(6) \quad \begin{aligned} \gamma \left(\frac{\partial^2 \nu}{\partial r^2} - \frac{\partial^2 \nu}{\partial z^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} - \frac{1}{r^2} \nu \right) + k \omega - 2k\nu &= \rho j \left(\frac{\partial \nu}{\partial t} + \nu \frac{\partial \nu}{\partial r} + w \frac{\partial \nu}{\partial z} \right), \\ \frac{\partial v}{\partial r} + \frac{\partial w}{\partial z} + \frac{v}{r} &= 0, \end{aligned}$$

where

$$\omega = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial r} \quad \text{and} \quad H = p + \frac{1}{2} \rho (v^2 + w^2).$$

The boundary conditions are: i)

$$(7) \quad v = 0, \quad w = 0, \quad \frac{\partial}{\partial r} \left(\frac{1}{r} \nu \right) = 0 \quad \text{on} \quad r = a(z);$$

ii) the flux across a cross-section of the tube is prescribed as

$$(8) \quad \int_0^{a(z)} dr \int_0^{2\pi} r w d\theta = 2\pi \psi_0 (1 + \kappa \cos \sigma t),$$

where ψ_0 is a constant, κ is the amplitude of the pulse which is assumed small and σ is the frequency of oscillation;

iii) since the solution has to be regular on the axis of the tube we have

$$(9) \quad v = 0, \quad \frac{\partial w}{\partial r} = 0, \quad \text{and} \quad \nu \text{ is finite at } r = 0.$$

We introduce the stream function $\psi(r, z, t)$ in the following manner :

$$(10) \quad v = \frac{1}{r} \frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r} \frac{\partial \psi}{\partial r}.$$

Eliminating function H from first two equations of (5) by using (10) we get

$$(11) \quad (\mu + k) \left(\omega_{rr} + \omega_{zz} + \frac{1}{r} \omega_r - \frac{1}{r^2} \omega \right) - k \left(v_{rr} + v_{zz} + \frac{1}{r} v_r - \frac{1}{r^2} v \right) =$$

$$= \rho \omega_t + \frac{\rho}{r} \left(\omega_r \psi_z - \omega_z \psi_r - \frac{1}{r} \omega \psi_z \right),$$

$$(12) \quad \gamma \left(v_{rr} + v_{zz} + \frac{1}{r} v_r - \frac{1}{r^2} v \right) + k v - 2k v = \rho j v_t + \frac{\rho j}{r} (v_r \psi_z - v_z \psi_r),$$

$$\omega = \frac{1}{r} \left(\psi_{zz} + \psi_{rr} - \frac{1}{r} \psi_r \right),$$

where the subscripts denote the partial differentiation with respect to the concerned variable and this notation is followed throughout the paper.

The boundary conditions (7) - (9) in terms of ψ can be written as

$$(13) \quad \psi_r = 0, \quad \psi_z = 0, \quad \psi = \psi_0(1 + \alpha \cos \sigma t), \quad \frac{\partial}{\partial r} \left(\frac{1}{r} v \right) = 0 \quad \text{on } r = a(z),$$

$$(14) \quad \left. \begin{aligned} \psi &= 0, \quad \frac{1}{r} \psi_z = 0, \\ \frac{1}{r} \psi_{rr} - \frac{1}{r^2} \psi_r &= 0, \quad v \text{ is finite} \end{aligned} \right\} \text{ on } r = 0.$$

The problem is to solve (11) and (12) with the boundary conditions (13) and (14).

The cross-section of the tube is assumed to vary slowly in the axial direction in which we take

$$(15) \quad a(z) = a_0 s \left(\frac{\varepsilon z}{a_0} \right), \quad (0 < \varepsilon \ll 1),$$

where a_0 is a constant characteristic radius of the tube and ε is a small parameter such as $\varepsilon \rightarrow 0$ corresponds to the tube of constant radius.

Introducing the following dimensionless variables

$$(16) \quad R = \frac{r}{a_0}, \quad Z = \varepsilon \frac{z}{a_0}, \quad T = \sigma t, \quad J = \frac{j}{a_0^2},$$

$$\Phi(R, Z, T) = \frac{\psi}{\psi_0}, \quad \Omega(R, Z, T) = \frac{\omega a_0^3}{\psi_0}, \quad U = \frac{v a_0^3}{\psi_0},$$

the governing equations and the conditions (11) to (14) become

$$\left(\frac{1}{R_e} + \frac{1}{R_k}\right) \left(\Omega_{RR} + \frac{1}{R} \Omega_R - \frac{1}{R^2} \Omega + \varepsilon^2 \Omega_{ZZ} \right) - \frac{1}{R_k} \left(U_{RR} + \right. \\ \left. + \frac{1}{R} U_R - \frac{1}{R^2} U + \varepsilon^2 U_{ZZ} \right) = m \Omega_T + \frac{\varepsilon}{R} \left(\Omega_R \Phi_Z - \Omega_Z \Phi_R - \frac{1}{R} \Omega \Phi_Z \right), \quad (17)$$

$$\left(\frac{1}{R_Y} \left(U_{RR} + \frac{1}{R} U_R - \frac{1}{R^2} U + \varepsilon^2 U_{ZZ} \right) + \frac{1}{R_k} \Omega - \frac{2}{R_k} U = \right. \\ \left. = m J U_T + J \frac{\varepsilon}{R} (U_R \Phi_Z - U_Z \Phi_R), \quad (18)$$

$$\Omega = \frac{1}{R} \left(\varepsilon^2 \Phi_{ZZ} + \Phi_{RR} - \frac{1}{R} \Phi_R \right), \quad (19)$$

$$\Phi_Z = \Phi_R = 0, \quad \Phi = 1 + \chi \cos T, \quad \frac{\partial}{\partial R} \left(\frac{1}{R} U \right) = 0, \quad \text{on } R = s(Z), \quad (20)$$

$$\left. \begin{aligned} \Phi = 0, \quad \frac{1}{R} \Phi_Z = 0, \\ \frac{1}{R} \Phi_{RR} - \frac{1}{R^2} \Phi_R = 0, \quad U \text{ is finite} \end{aligned} \right\} \text{ on } R = 0,$$

where

$$R_e = \frac{\rho \psi_0}{\mu a_0}, \quad R_k = \frac{\rho \psi_0}{k a_0}, \quad R_Y = \frac{\rho \psi_0 a_0}{\gamma},$$

are the Reynolds numbers of the flow which are considered very small, and $m = \sigma a_0^2 / \psi_0$ is a nondimensional number.

3. Solutions of the Equations. We shall take the pulsatile flow consisting of a steady flow and an oscillatory flow which may be considered very small such that terms of the order of χ^2 are neglected in the analysis. The solutions of (17) and (18) are taken in the form

$$\begin{aligned} \Omega &= \Omega^0 + \varepsilon \Omega^1 + \chi (\bar{\Omega}^0 + \varepsilon \bar{\Omega}^1) e^{i T} + \dots, \\ U &= U^0 + \varepsilon U^1 + \chi (\bar{U}^0 + \varepsilon \bar{U}^1) e^{i T} + \dots, \\ \Phi &= \Phi^0 + \varepsilon \Phi^1 + \chi (\bar{\Phi}^0 + \varepsilon \bar{\Phi}^1) e^{i T} + \dots \end{aligned} \quad (21)$$

Substituting (21) in (17) to (20) and equating the coefficients of $e^{i T}$ and powers of ε , we obtain the following coupled differential equations

$$\left\{ \begin{aligned} \Omega^0 &= \frac{1}{R} \left(\Phi_{RR}^0 - \frac{1}{R} \Phi_R^0 \right), \\ \left(\frac{1}{R_e} + \frac{1}{R_k} \right) A_R \Omega^0 - \frac{1}{R_k} A_R U^0 &= 0, \\ \frac{1}{R_Y} A_R U^0 - \frac{2}{R_k} U^0 + \frac{1}{R_k} \Omega^0 &= 0; \end{aligned} \right. \quad (22)$$

$$(23) \quad \begin{cases} \bar{\Omega}^0 = \frac{1}{R} \left(\bar{\Phi}_{RR}^0 - \frac{1}{R} \bar{\Phi}_R^0 \right), \\ \left(\frac{1}{R_e} + \frac{1}{R_k} \right) A_R \bar{\Omega}^0 - \frac{1}{R_k} A_R \bar{U}^0 = m i \bar{\Omega}^0, \\ \frac{1}{R_Y} A_R \bar{U}^0 - \frac{2}{R_k} \bar{U}^0 + \frac{1}{R_k} \bar{\Omega}^0 = m J i \bar{U}^0; \end{cases}$$

$$(24) \quad \begin{cases} \Omega^1 = \frac{1}{R} \left(\Phi_{RR}^1 - \frac{1}{R} \Phi_R^1 \right), \\ \left(\frac{1}{R_e} + \frac{1}{R_k} \right) A_R \Omega^1 - \frac{1}{R_k} A_R U^1 = \frac{1}{R} \left[\Phi_Z^0 \left(\Omega_R^0 - \frac{1}{R} \Omega^0 \right) - \Omega_Z^0 \Phi_R^0 \right], \\ \frac{1}{R_Y} A_R U^1 - \frac{2}{R_k} U^1 + \frac{1}{R_k} \Omega^1 = \frac{J}{R} \left(\Phi_Z^0 U_R^0 - U_Z^0 \Phi_R^0 \right); \end{cases}$$

$$(25) \quad \begin{cases} \bar{\Omega}^1 = \frac{1}{R} \left(\bar{\Phi}_{RR}^1 - \frac{1}{R} \bar{\Phi}_R^1 \right), \\ \left(\frac{1}{R_e} + \frac{1}{R_k} \right) A_R \bar{\Omega}^1 - \frac{1}{R_k} A_R \bar{U}^1 = m i \bar{\Omega}^1 + \frac{1}{R} \bar{\Phi}_Z^0 \left(\Omega_R^0 - \frac{1}{R} \Omega^0 \right) + \\ + \frac{1}{R} \left[\Phi_Z^0 \left(\bar{\Omega}_R^0 - \frac{1}{R} \bar{\Omega}^0 \right) - \Omega_Z^0 \bar{\Phi}_R^0 - \bar{\Omega}_Z^0 \Phi_R^0 \right], \\ \frac{1}{R_Y} A_R \bar{U}^1 - \frac{2}{R_k} \bar{U}^1 + \frac{1}{R_k} \bar{\Omega}^1 = J m i \bar{U}^1 + \\ + J \frac{1}{R} (\bar{\Phi}_Z^0 U_R^0 + \Phi_Z^0 \bar{U}_R^0 - \bar{\Phi}_R^0 U_Z^0 - \Phi_R^0 \bar{U}_Z^0), \end{cases}$$

where

$$A_R = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} - \frac{1}{R^2},$$

and the corresponding boundary conditions are

$$(26) \quad \left. \begin{aligned} \Phi_R = \bar{\Phi}_R^\alpha = \Phi_Z^\alpha = \bar{\Phi}_Z^\alpha = 0, \quad (\alpha=0, 1) \\ \Phi^0 = \bar{\Phi}^0 = 1, \quad \Phi^1 = \bar{\Phi}^1 = 0, \\ \frac{\partial}{\partial R} \left(\frac{1}{R} U^\alpha \right) = \frac{\partial}{\partial R} \left(\frac{1}{R} \bar{U}^\alpha \right) = 0, \quad (\alpha=0, 1) \end{aligned} \right\} \text{ on } R=s(Z),$$

$$(27) \quad \left. \begin{aligned} \Phi^\alpha - \bar{\Phi}_R^\alpha &= 0, \quad (\alpha = 0, 1) \\ \frac{1}{R} \Phi_Z^\alpha &= \frac{1}{R} \bar{\Phi}_Z^\alpha = 0, \quad (\alpha = 0, 1) \\ \frac{\partial}{\partial R} \left(\frac{1}{R} \Phi_R^\alpha \right) &= \frac{\partial}{\partial R} \left(\frac{1}{R} \bar{\Phi}_R^\alpha \right) = 0, \quad (\alpha = 0, 1) \\ U^\alpha, \bar{U}^\alpha &\text{ are finite, } (\alpha = 0, 1) \end{aligned} \right\} \text{ on } R = 0.$$

In order to reduce the length of the paper we shall not present the expressions of the solution. They will be discussed in graphs.

The shear stress at the wall of the tube $r = a(z)$ is

$$(28) \quad T_s = (t_{rr} - t_{zz}) \frac{a_z}{1 + a_z^2} + (t_{rz} - t_{zr} a_z^2) \frac{1}{1 + a_z^2},$$

where t_{rr} , t_{zz} , t_{rz} , t_{zr} are the components with the usual meaning.

The boundary conditions (13) will give rise to the following conditions

$$(29) \quad a_z \psi_{rr} + \psi_{rz} = 0, \quad \psi_{zz} + a_z \psi_{zr} = 0, \quad \text{on } r = a(z).$$

Using the above conditions in (28), we obtain the stress as

$$(30) \quad T_s = -(\mu + k)\omega + k\nu, \quad \text{on } r = a(z).$$

Introducing a dimensionless stress τ in the form

$$\tau = \frac{a_0^4}{\rho \psi_0^2} T_s,$$

the normalised shear stress at the wall is given by

$$(31) \quad \tau = -\left(\frac{1}{R_e} + \frac{1}{R_k}\right)\Omega + \frac{1}{R_k} U.$$

4. Discussion of the Results. The shear stress distribution on the walls of the tubes of two special cases of practical interest, namely tapered tube and peristaltic tube can be evaluated using (31).

The tapered tube geometry is an approximation for dichotomously branching vascular bed and the motion in such tube models would explain the blood flows in vascular beds. The equation to the wall of the tapered tube is given by

$$(32) \quad s = 1 + \alpha Z,$$

where α is the parameter characterising the tapered geometry of the tube. The behaviour of the shear stress on the wall for $\alpha = 0$ (with $T = 0$) when $\alpha = \pm 0.1$ for the case of viscous fluids and micropolar fluids is shown in Fig. 1. It is observed that the shearing stress would vanish at a larger distance from the origin. Moreover the shearing stress of micropolar fluid decrease and increase faster than that of viscous fluids. It is interesting to

notice that points of separation for $\alpha = \pm 0.1$ lie in the opposite regions from the origin. Thus the governing parameters causing separation could be the amplitude of the pulse, time and geometry of the tube.

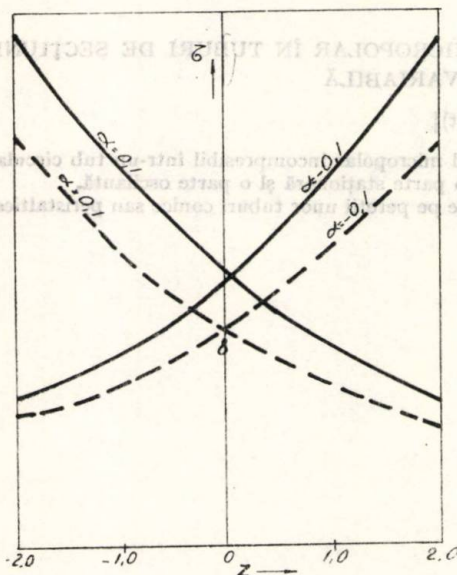


Fig. 1 — Shear stress for a tapered tube at $T=0, \alpha=0$: —micropolar fluid; ---viscous fluid.

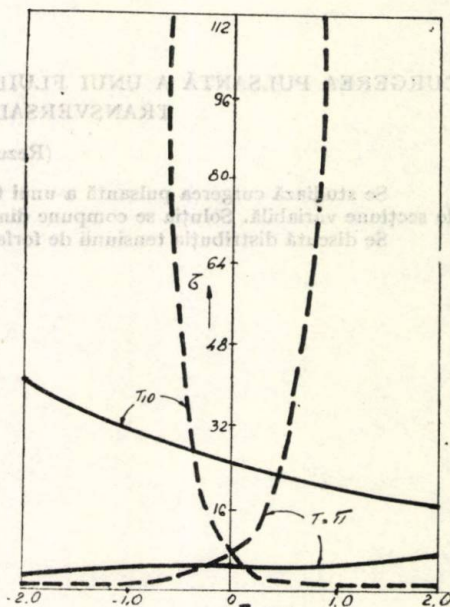


Fig. 2 — Shear stress for a peristaltic tube at different times ($\alpha=0$): —micropolar fluid; ---viscous fluid.

The peristaltic tube geometry is of great importance in physiology. The equation to the wall of the tube is

$$(33) \quad s = 1 + \sin(Z - T).$$

Shear stress distribution on the wall for this geometry at different times in the case of viscous fluids and micropolar fluids is shown in Fig. 2. As compared to the viscous fluids, for micropolar fluids very high values of shearing stress are not attained.

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CURGEREA PULSANTĂ A UNUI FLUID MICROPOLAR ÎN TUBURI DE SECȚIUNE TRANSVERSALĂ VARIABILĂ

(Rezumat)

Se studiază curgerea pulsantă a unui fluid micropolar incompresibil într-un tub circular de secțiune variabilă. Soluția se compune dintr-o parte staționară și o parte oscilantă.

Se discută distribuția tensiunii de forfecare pe pereții unor tuburi conice sau peristaltice.

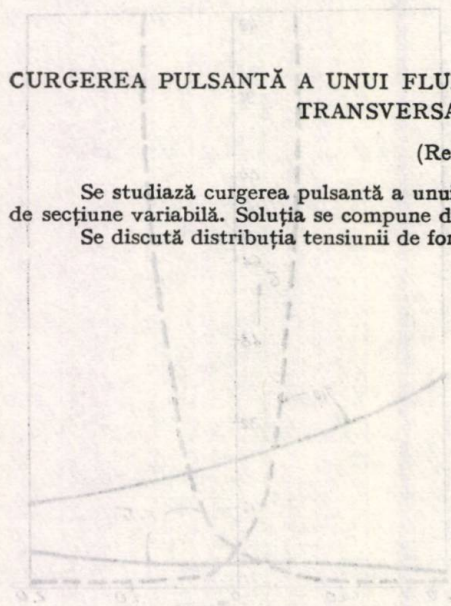


Fig. 2 — Shear stress for a peristaltic tube at different times $(\tau = 0)$; — micropolar fluid; — viscous fluid.

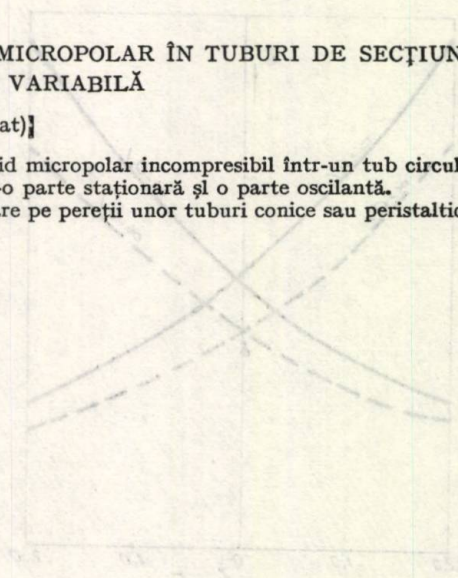


Fig. 3 — Shear stress for a tapered tube at $\tau = 0$; — micropolar fluid; — viscous fluid.

The peristaltic tube geometry is of great importance in physiology. The variation to the wall of the tube is

$$r = 1 + \sin(2\pi - t) \quad (3)$$

Shear stress distribution on the wall for the geometry is different times in the case of viscous fluids and micropolar fluids is shown in Fig. 2. As compared to the viscous fluids for micropolar fluids very high values of shear stress are not attained.

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PLANE BOUNDARY-VALUE PROBLEMS IN THE NONLINEAR GENERALIZED THEORY OF RODS

I. DISPLACEMENT EQUILIBRIUM EQUATIONS

BY

GH. CHIORESCU

1. Introduction. The present paper and a companion one (part II) are concerned with some nonlinear problems in the generalized theory of elastic rods. Starting with the theory formulated in [1] — Part I, we derive the displacement equilibrium equations for any plane non-linear boundary-value problems. Part II is devoted to approximation of the solution of a such problem by using a Newton-Kantorovich method.

2. Basic Equations. A generalized rod is a body $\mathcal{R}$ comprising a material line and two deformable vectors (called directors) attached to every point of the line. Set the particles of the material line of the body $\mathcal{R}$ be identified with the coordinate S . In a reference configuration of $\mathcal{R}$, let the reference line be referred to by C , let $\mathbf{R}$ be the position vector of C and $\mathbf{D}_\alpha$ ($\alpha = 1, 2$), the reference values of the two directors. The coordinate S is the arc length parameter of C and, from this point of view, differs to coordinate ξ used in [1]. Let the line occupied by the material line of $\mathcal{R}$ in the present configuration at time t be referred to by c , let $\mathbf{r}$ be the position vector of c and $\mathbf{d}_\alpha$ the directors at $\mathbf{r}$. In the following, the vectors and the tensor are referred to the following base vectors: $\mathbf{a}_3 = d\mathbf{r}/dS$, the unit principal normal $\mathbf{a}_1$ and unit binormal $\mathbf{a}_2$.

We summarize now the basic equations of the theory [1].

i) Geometrical equations

$$(2.1) \quad \begin{aligned} \gamma_{ij} &= h_{ij} - H_{ij} = \mathbf{d}_i \cdot \mathbf{d}_j - \mathbf{D}_i \cdot \mathbf{D}_j, \\ k_{\alpha i} &= \lambda_{\alpha i} - \Lambda_{\alpha i} = \mathbf{d}'_\alpha \cdot \mathbf{d}_i - \mathbf{D}'_\alpha \cdot \mathbf{D}_i, \end{aligned}$$

where $\mathbf{d}_3 = \mathbf{a}_3$, $\mathbf{D}_3 = \mathbf{A}_3 = d\mathbf{R}/dS$, and prime denotes differentiation with respect to S .

ii) Equilibrium equations

$$(2.2) \quad \begin{aligned} \mathbf{n}' + \rho_0 \mathbf{f} &= 0, \\ \mathbf{p}^\alpha + \rho_0 \mathbf{l}^\alpha &= \pi^\alpha, \end{aligned}$$

where ρ_0 is the mass density in the reference configuration, $\mathbf{f}$ is the body

force, $\mathbf{l}^\alpha$ are the director couples, $\mathbf{n}$ is the contact force, $\mathbf{p}^\alpha$ are the contact director couples and π^α are the intrinsic director couples.

i ii) Constitutive equations

$$(2.3) \quad \bar{\pi} = 2\rho_0 \frac{\partial \Psi^*}{\partial \gamma_{33}}, \quad \bar{\pi}^{(\alpha\beta)} = 2\rho_0 \frac{\partial \Psi^*}{\partial \gamma_{\alpha\beta}}, \quad \bar{\pi}^\alpha = 2\rho_0 \frac{\partial \Psi^*}{\partial \gamma_{\alpha 3}}, \quad p^{\alpha i} = \rho_0 d_j^i \frac{\partial \Psi^*}{\partial k_{\alpha j}},$$

$$(2.4) \quad \bar{\pi} = d^3_i (n^i - \lambda_{\nu^3} p^{\nu i}), \quad \pi^\alpha = 2(d^{\alpha i} n^i - d^3_i \lambda_{\nu^3} p^{\nu i}),$$

$$(2.5) \quad 2\bar{\pi}^{(\alpha\beta)} = d^{\beta i} (\pi^{\alpha i} - \lambda_{\nu^3} p^{\nu i}) + d^{\alpha i} (\pi^{\beta i} - \lambda_{\nu^3} p^{\nu i}),$$

$$(2.5) \quad \mathbf{a}_3 \times (\mathbf{n} + \lambda_{\alpha^3} \mathbf{p}^\alpha) + \mathbf{d}_\alpha \times (\pi^\alpha + \lambda_{\beta^3} \mathbf{p}^\beta) = 0,$$

where Ψ^* is the specific Helmholtz free energy.

3. Displacement Equilibrium Equations of the Plane Boundary-value Problems. We study the following deformations: longitudinal extension, the transversal contraction, the bending and the shear. In this end, we suppose the present configuration to be given by

$$(3.1) \quad \mathbf{r} = \mathbf{r}(S), \quad \mathbf{d}_1 = (1 + \rho) \left(\cos \beta \mathbf{a}_1 + \frac{\sin \beta}{1 + \varepsilon} \mathbf{a}_3 \right) \quad \text{and} \quad \mathbf{d}_2 = \mathbf{a}_2,$$

where $\varepsilon = a_{33}^{1/2} - 1$ and $\rho = (\mathbf{d}_1 \cdot \mathbf{d}_1)^{1/2} - 1$ are relativ deformation measures, β is the angle between $\mathbf{a}_1$ and $\mathbf{d}_1$ and Φ is the angle between $\mathbf{d}_1$ and Oy -axis. Further the directors $\mathbf{d}_\alpha$ satisfy the restriction $[\mathbf{d}_1 \mathbf{d}_2 \mathbf{a}_3] > 0$, which ensures that $\mathbf{d}_\alpha$ are nowhere tangential to c and that $\mathbf{d}_1, \mathbf{d}_2$ never change their relative orientation with respect to each other and $\mathbf{a}_3$. The reference configuration is given by

$$(3.2) \quad \mathbf{R} = \mathbf{R}(S), \quad \mathbf{D}_1 = \mathbf{A}_1 \quad \text{and} \quad \mathbf{D}_2 = \mathbf{A}_2.$$

Without loss of generality we assume that the value of β in the reference configuration is 0. We have also the relations

$$(3.3) \quad \theta = \beta + \Phi \quad \text{and} \quad s' = a_{33}^{1/2},$$

where s is the arc length parameter of the curve c . Let Θ denotes the value of θ in the reference configuration.

In order to derive the deformation measures (2.1) we need the following equations;

$$(3.4) \quad \mathbf{a}'_1 = -\frac{\theta'}{1 + \varepsilon} \mathbf{a}_3, \quad \mathbf{a}'_2 = 0, \quad \mathbf{a}'_3 = \theta'(1 + \varepsilon) \mathbf{a}_1 + \frac{\varepsilon'}{1 + \varepsilon} \mathbf{a}_3,$$

$$(3.5) \quad \mathbf{d}^1 = \frac{1}{(1 + \rho) \cos \beta} \mathbf{a}_1, \quad \mathbf{d}^2 = \mathbf{a}_2, \quad \mathbf{d}^3 = -\frac{\sin \beta}{(1 + \varepsilon) \cos \beta} \mathbf{a}_1 + \frac{1}{(1 + \varepsilon)^2} \mathbf{a}_3,$$

$$(3.6) \quad \mathbf{a}^1 = \mathbf{a}_1, \quad \mathbf{a}^2 = \mathbf{a}_2, \quad \mathbf{a}^3 = \frac{\mathbf{a}_3}{(1 + \varepsilon)^2},$$

$$(3.7) \quad \lambda_1^1 = \frac{\rho'}{1 + \rho} + \Phi' \frac{\sin \beta}{\cos \beta}, \quad \lambda_1^3 = -\Phi' \frac{(1 + \rho)}{(1 + \varepsilon) \cos \beta},$$

and all the remaining λ_j^i vanish. Thus the relative measures (2.1) are given by

$$(3.8) \quad \gamma_{11} = (1 + \rho)^2 - 1, \quad \gamma_{13} = (1 + \rho)(1 + \varepsilon) \sin \beta, \quad \gamma_{33} = (1 + \varepsilon)^2 - 1, \\ k_{11} = \rho'(1 + \rho), \quad k_{13} = \rho'(1 + \varepsilon) \sin \beta - (1 + \rho)(1 + \varepsilon) \Phi' \cos \beta + \Theta',$$

and all the remaining measures vanish. If we set

$$(3.9) \quad \xi = (1 + \varepsilon) \cos \beta - 1, \quad \eta = (1 + \varepsilon) \sin \beta,$$

the equations (3.8) become

$$(3.10) \quad \gamma_{11} = (1 + \rho)^2 - 1, \quad \gamma_{13} = (1 + \rho)\eta, \quad \gamma_{33} = (1 + \xi)^2 + \eta^2 - 1, \\ k_{11} = \rho'(1 + \rho), \quad k_{13} = \rho'\eta - \Phi'(1 + \rho)(1 + \xi) + \Theta'.$$

Observation 3.1. The measure (3.10) are not identical with those of A n t m a n [2], because we do not use the same base vectors.

Observation 3.2. If we adopt the Bernoulli's hypothesis than β must be zero.

Supposing that

$$(3.11) \quad \bar{\pi}^{(12)} = \bar{\pi}^{(22)} = \bar{\pi}^{(2)} = p^{21} = p^{22} = p^{23} = p^{12} = 0,$$

we obtain the following constitutive equations

$$(3.12) \quad \bar{\pi} = 2\rho_0 \frac{\partial \Psi^*}{\partial \gamma_{33}}, \quad \bar{\pi}^{(11)} = 2\rho_0 \frac{\partial \Psi^*}{\partial \gamma_{11}}, \quad \bar{\pi}^1 = 2\rho_0 \frac{\partial \Psi^*}{\partial \gamma_{13}}, \\ p^{11} = \rho_0(1 + \rho) \cos \beta \frac{\partial \Psi^*}{\partial k_{11}}, \quad p^{13} = \rho_0 \frac{(1 + \rho) \sin \beta}{1 + \varepsilon} \frac{\partial \Psi^*}{\partial k_{11}} + \frac{\partial \Psi^*}{\partial k_{13}}.$$

By using (2.4) and (2.5) it results that

$$(3.13) \quad \pi^{22} = n^2 = \pi^{12} = \pi^{21} = \pi^{23} = 0,$$

$$(3.14) \quad \bar{\pi} = d^3_1(n^1 - \lambda_1^3 p^{11}) + d^3_3(n^3 - \lambda_1^3 p^{13}), \\ \bar{\pi}^{(1)} = 2(d^1_1 n^1 - d^3_1 \lambda_1^1 p^{11} - d^3_3 \lambda_1^1 p^{13}), \\ 2\bar{\pi}^{(11)} = 2d^1_1(\pi^{11} - \lambda_1^1 p^{11}), \\ n^1 + \lambda_1^3 p^{11} + d^1_3 \pi^{11} + d^1_3 \lambda_1^1 p^{11} - d^1_1 \pi^{13} - d^1_1 \lambda_1^1 p^{13} = 0.$$

From (3.12), (3.13) it follows that body forces must satisfy the restriction

$$(3.15) \quad f^2 = l^{12} = l^{21} = l^{22} = l^{23} = 0.$$

Thus, in terms of tensor components the equilibrium equations are

$$(3.16) \quad (\eta^1)' + \eta^3 \theta' (1 + \varepsilon) + \rho_0 f^1 = 0, \\ (\eta^3)' - \eta^1 \frac{\theta'}{1 + \varepsilon} + \eta^3 \frac{\varepsilon'}{1 + \varepsilon} + \rho_0 f^3 = 0, \\ (p^{11})' + p^{13} \theta' (1 + \varepsilon) + \rho_0 l^{11} = \pi^{11}, \\ (p^{13})' - p^{11} \frac{\theta'}{1 + \varepsilon} + p^{13} \frac{\varepsilon'}{1 + \varepsilon} + \rho_0 l^{13} = \pi^{13}.$$

We now proceed to obtain the constitutive relations in terms of a response function w defined by

$$(3.17) \quad w(\rho, \xi, \eta, \rho', \Phi', S) = \Psi^*(\gamma_{11}, \gamma_{13}, \gamma_{33}, k_{11}, k_{13}, S).$$

By solving (3.14) in respect to η^1 , η^3 , π^{11} , and π^{13} expressing the derivatives of w with the derivatives of ρ and using (3.12) we get the following constitutive equations:

$$(3.18) \quad \begin{aligned} \pi^{11} &= \rho_0 A \left[(1 + \xi) w_\rho - \frac{\eta(1 + \xi)}{1 + \rho} w_\eta + \frac{\eta^2}{1 + \rho} w_\xi + \Phi' \eta w_{\rho'} - \right. \\ &\quad \left. - \frac{1 + \xi}{1 + \rho} \left(\frac{\eta \rho'}{(1 + \rho)(1 + \xi)} + \Phi' \right) w_{\Phi'} \right], \\ \eta^1 &= \rho_0 A [(1 + \xi) w_\eta - \eta w_\xi], \quad \eta^3 = \rho_0 A \left[\left(\frac{1}{(1 + \xi)A} - \frac{\eta^2}{1 + \xi} \right) w_\xi + \eta A w_\eta \right], \\ \pi^{13} &= \rho_0 A \left\{ \eta A w_\rho + \left(\frac{1}{A} - \eta^2 A \right) w_\eta + \frac{1}{(1 + \rho)(1 + \xi)} \left(\frac{-\eta}{A} + \eta^3 A w_\xi \right) + \right. \\ &\quad \left. + \left(\frac{-\rho' A}{1 + \rho} + (1 + \rho) \eta (1 + \xi) A^2 - \frac{\Phi'}{1 + \xi} \right) w_{\rho'} + \right. \\ &\quad \left. + \left[\frac{2}{A(1 + \rho)(1 + \xi)} \left(\frac{\rho'}{1 + \rho} + \frac{\Phi' \eta}{1 + \rho} \right) - \frac{\eta A}{1 + \rho} \left(\frac{\eta^2 \Phi'}{(1 + \rho)^2} + \frac{2\eta \rho'}{(1 + \rho)(1 + \xi)} + \Phi' \right) + \right. \right. \\ &\quad \left. \left. + \eta \left(\eta A^2 - \frac{\Phi'}{A(1 + \rho)(1 + \xi)^2} - 1 \right) \right] w_{\Phi'} \right\}, \quad p^{11} = \rho_0 A [(1 + \xi) w_\rho + \eta w_{\Phi'}], \\ p^{13} &= \rho_0 A \left[\eta A w_{\rho'} + \left(\frac{\eta A}{(1 + \rho)(1 + \xi)} - \frac{1}{A(1 + \rho)(1 + \xi)} \right) w_{\Phi'} \right], \end{aligned}$$

where A denotes $1/\sqrt{\eta^2 + (1 + \xi)^2}$.

Introducing (3.18) and using (3.9) we obtain the following displacement equilibrium equations:

$$(3.19)_1 \quad \begin{aligned} &\xi \left[(1 + \xi) w_{\eta\xi} - \eta w_{\xi\xi} + (1 - (1 + \xi)A^2 + \eta A) w_\eta + \left((1 + \xi)A + \frac{\eta A}{1 + \xi} - \right. \right. \\ &\quad \left. \left. - \frac{\eta^3 A^3}{1 + \xi} \right) w_\xi \right] + \eta' [(1 + \xi) w_{\eta\eta} - \eta w_{\xi\eta} + (\eta A - \eta A^2) w_\eta + (1 + A^2 - \\ &\quad - \eta^2 A^3) w_\xi] + \rho'' [(1 + \xi) w_{\rho\rho} - \eta w_{\xi\rho}] + \Phi'' [(1 + \xi) w_{\eta\Phi} - \eta w_{\xi\Phi}] + \\ &\quad + (1 + \xi) \rho' w_{\eta\rho} + (1 + \xi) w_{\eta S} - \eta \rho' w_{\xi\rho} - \eta w_{\xi S} + \frac{\rho_0'}{\rho_0} \left((1 + \xi) + \Phi' \eta A \right) w_\eta + \\ &\quad + \left(\frac{\Phi'}{A(1 + \xi)} - \frac{\eta^2 \Phi'}{1 + \xi} A - \frac{\rho_0'}{\rho_0} \eta \right) w_\xi + \frac{1}{A} f^1 = 0, \end{aligned}$$

$$\begin{aligned}
 & \xi'[(1+\xi)w_{\xi\xi} + ((-\eta^2 + (1+\xi)^2 A^2 - \eta^2 A^2 + (1+\xi)^2 A^3)w_{\xi} + \eta w_{\eta\xi} + \\
 & + (-\eta(1+\xi)A^2 + \eta(1+\xi)A^3)w_{\eta} + \eta'[(1+\xi)w_{\xi\eta} + \eta w_{\eta\eta} + \\
 & + (-\eta(1+\xi)A^2 + \eta(1+\xi)A^3)w_{\xi} + ((-\eta^2 + (1+\xi)^2 A^2 - (1+\xi)^2 A^3 + \\
 (3.19)_2 & + \eta^2 A^3)w_{\eta} + \rho'' \left[\frac{1+\xi}{A} w_{\xi\rho} + \eta w_{\eta\rho} \right] + \Phi''[(1+\xi)w_{\xi\Phi} + \eta w_{\eta\Phi}] + \\
 & + \left(\frac{\rho'_0}{\rho_0} (1+\xi) + \Phi' \eta \right) w_{\xi} + \left(\frac{\rho'_0}{\rho_0} \eta - \Phi' (1+\xi) \right) w_{\eta} + \rho' (1+\xi) w_{\xi\rho} + \\
 & + \rho' \eta w_{\eta\rho} (1+\xi) w_{\xi s} + \eta A^2 w_{\eta s} + \frac{1}{A^2} f^3 = 0,
 \end{aligned}$$

$$\begin{aligned}
 & \xi'(1+\xi)w_{\rho\xi} + \eta^2 A^2 w_{\rho} + \eta w_{\Phi\xi} - \eta(1+\xi)A^2 w_{\Phi} + \\
 & + \eta'[(1+\xi)w_{\rho\eta} - \eta(1+\xi)A^2 w_{\rho} + \eta w_{\Phi\eta} + (1+\xi)^2 A w_{\Phi}] + \\
 & + \rho''[(1+\xi)w_{\rho\rho} + \eta w_{\Phi\rho}] + \Phi''[(1+\xi)w_{\rho\Phi} + \eta w_{\Phi\Phi}] + \\
 (3.19)_3 & + (1+\xi)\rho' w_{\rho\rho} + (1+\xi)w_{\rho s} + \left(\frac{\rho'_0}{\rho_0} (1+\xi) + \eta A - \Phi' \eta \right) w_{\rho} + \eta \rho' w_{\Phi\rho} +
 \end{aligned}$$

$$\begin{aligned}
 & + \eta w_{\Phi s} + \left(\frac{\rho'_0}{\rho_0} \eta + (\eta A^2 - 1) \right) \frac{1}{A(1+\rho)(1+\xi)} + \frac{\eta \rho'}{(1+\rho)^2} + \\
 & + \frac{\Phi'(1+\xi)}{1+\rho} \left(w_{\Phi} + \frac{\eta(1+\xi)}{1+\rho} w_{\eta} - (1+\xi)w_{\rho} - \frac{\eta^2}{1+\rho} + \frac{1}{A} l^{11} \right) = 0,
 \end{aligned}$$

$$\begin{aligned}
 & \xi'[\eta w_{\rho\xi} + \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi\xi} + \left(\eta A^2 - \frac{\eta}{(1+\rho)(1+\xi)^2} + \right. \\
 & + \frac{1}{(1+\rho)(1+\xi)^2} A^2 \frac{\eta A^2}{1+\rho} - \frac{1}{1+\rho} \left. \right) w_{\Phi} + \eta' \left[\eta w_{\rho\eta} + \right. \\
 & + \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi\eta} + \left(A - \frac{(1+\xi)^2 - \eta^2}{(1+\rho)(1+\xi)} - \eta(1+\xi)A^2 + \right. \\
 (3.19)_4 & + \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi} \left. \right] + \rho'' \left[\eta w_{\rho\rho} + \right. \\
 & + \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi\rho} \left. \right] + \Phi'' \left[\eta w_{\rho\Phi} + \right. \\
 & + \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi\Phi} \left. \right] + \rho' \eta w_{\rho\rho} + \eta w_{\rho s} + \\
 & + \frac{\rho'}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi\rho} + \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) w_{\Phi s} +
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{\rho'_0}{\rho_0} \eta - \Phi'(1+\xi) \frac{\rho'_0 \eta}{1+\rho} - (1+\rho)(1+\xi)\eta A + \frac{\Phi'}{(1+\xi)A^2} \right) w_{\rho'} + \\
 & + \left[\frac{\rho'_0}{\rho_0} \frac{1}{(1+\rho)(1+\xi)} \left(\eta - \frac{1}{A^2} \right) - \Phi' \eta - \frac{2}{(1+\rho)(1+\xi)A^2} \left(\frac{\rho'}{1+\rho} + \right. \right. \\
 & \left. \left. + \frac{\Phi' \eta}{1+\rho} \right) - \frac{\eta}{1+\rho} \left(\frac{\eta^2 \Phi'}{(1+\rho)^2} + \frac{2\eta \rho'}{(1+\rho)(1+\xi)} + \Phi' \right) + \right. \\
 & \left. + \eta \left(\eta A - \frac{\Phi'}{(1+\rho)(1+\xi)^2} A^2 - \frac{1}{A} \right) \right] w_{\Phi} + \frac{1}{A^2} + l^{13} = 0.
 \end{aligned}
 \tag{3.19}$$

Thus we have four nonlinear ordinary differential equations in four unknowns $\varepsilon, \xi, \eta, \rho$. In order to write the system (3.19), the function Ψ^* , the mass density ρ_0 , the reference configuration (Θ) , and the body forces f^1, f^3, l^{11} and l^{13} must be given. Of course some boundary conditions must be also attached.

In the Part II we shall solve a such boundary-value problem by the iterative Newton's method.

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PROBLEME LA LIMITĂ PLANE ÎN TEORIA NELINIARĂ GENERALIZATĂ A BARELOR

I. Ecuațiile de echilibru în deplasări

(Rezumat)

În această lucrare (Partea I-a a unei lucrări mai mari) se studiază ecuațiile de bază ale teoriei neliniare generalizate a barelor pentru probleme plane. Folosind teoria din [1] se stabilesc ecuațiile de echilibru în deplasări pentru orice deformare plană, orice fel de material (elastic) și orice fel de forțe masice. Astfel de ecuații în deplasări au fost stabilite în [1], pentru flexiunea unei bare izotrope cu secțiune circulară, ecuații care s-au redus la ecuațiile clasice ale lui Timošenko cînd raza secțiunii transversale este constantă.

THEOREM OF EXISTENCE AND UNIQUENESS IN RELATIVISTIC RADIATIVE HYDRODYNAMICS

BY

GH. PROCOPIUC

1. Introduction. The main point of this paper is to give a theorem of existence and uniqueness of the solution of Cauchy problem for the equations of relativistic radiative hydrodynamics, by extending the results of Fourès-Bruhat [1] and Lichnerowicz [2], [3] to the case when the fluid motion is made in the presence of a radiation field [4]. To this effect we use a result of Leray relating to the hyperbolic differential equations [5].

2. The Equations of Relativistic Radiative Hydrodynamics. We consider a relativistic thermodynamical perfect fluid in the presence of a radiation field. The energy-momentum tensor is here the sum of the dynamic energy-momentum tensor of the fluid and of the energy-momentum tensor of the radiation field:

$$(2.1) \quad T_{\alpha\beta} = (\rho u_{\alpha} u_{\beta} - p g_{\alpha\beta}) + \tau_{\alpha\beta}^{(r)},$$

where ρ is the proper matter density of the fluid, p its pressure, u_{α} the unitary velocity,

$$(2.2) \quad g^{\alpha\beta} u_{\alpha} u_{\beta} = 1$$

and f is the index of the fluid, which is given by

$$(2.3) \quad f = c^2 \left(1 + \frac{i}{c^2} \right),$$

where i is the specific enthalpy.

According to [4], the part of energy-momentum tensor corresponding to the radiation field can be written

$$(2.4) \quad \tau_{\alpha\beta}^{(r)} = E^{(r)} u_{\alpha} u_{\beta} + (u_{\alpha} u_{\beta} - g_{\alpha\beta}) p^{(r)} + S_{\alpha} u_{\beta} + S_{\beta} u_{\alpha},$$

where $E^{(r)}$ is the energy density of radiation field, $p^{(r)} = \frac{1}{3} E^{(r)}$ the radiation pressure and S^{α} the radiative flux corresponding to u_{α} .

Since

$$T_{\alpha\beta}u^\beta = [rf - p + E^{(r)}]u_\alpha + S_\alpha$$

it is clear that $rf - p + E^{(r)}$ is the total energy density of the fluid. Then, the coefficient of the $g_{\alpha\beta}$ in (2.1) with $\tau_{\alpha\beta}^{(r)}$ given by (2.4), $p + p^{(r)}$ is the total pressure.

If T is the proper temperature of the fluid and S its specific proper entropy, we have, as in classical hydrodynamics

$$TdS = di - \frac{1}{r} dp$$

and thus, with (2.3)

$$(2.5) \quad dp = r df - r T dS.$$

We assume that r is a given function of the thermodynamical variables f and S , $r = r(f, S)$, the form of this function depending on the internal structure of the fluid.

The main equations of relativistic radiative hydrodynamics are the Einstein equations

$$(2.6) \quad S_{\alpha\beta} = \chi T_{\alpha\beta}, \quad \chi = \text{const.}$$

where the Einstein tensor

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R,$$

with $R_{\alpha\beta}$ the Ricci tensor, is a conservative tensor

$$(2.7) \quad \nabla_\alpha S^{\alpha\beta} = 0$$

and the radiation field equations

$$(2.8) \quad \nabla_\alpha \tau^{(\alpha)\beta} = F^{(\beta)},$$

where

$$(2.9) \quad F_\alpha^{(r)} = -\sigma'_\alpha [S_\alpha + (E^{(r)} - aT^4)u_\alpha],$$

with σ'_α the absorption coefficient and a the radiation constant.

The equations of conservation (2.7) give

$$(2.10) \quad \nabla_\alpha T_\beta^\alpha \equiv \nabla_\alpha (rf u^\alpha) u_\beta + rf u^\alpha \nabla_\alpha u_\beta - \partial_\beta p + F_\beta^{(r)} = 0,$$

which, by contracted multiplying with u^β , become

$$f \nabla_\alpha (r u^\alpha) + r T u^\alpha \partial_\alpha S - \tau'_\alpha (E^{(r)} - aT^4) = 0.$$

In what follows we assume that the motion of the fluid is locally adiabatic, that is

$$(2.11) \quad u^\alpha \partial_\alpha S = 0$$

and then we obtain the equation of continuity

$$(2.12) \quad f \nabla_\alpha (r u^\alpha) = \sigma'_\alpha (E^{(r)} - aT^4).$$

From (2.10), with (2.12) and (2.5), it follows the differential system of the stream lines of the fluid

$$(2.13) \quad rfu^\alpha \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta) r(\partial_\alpha f - T \partial_\alpha S) = \sigma'_\alpha S^\beta.$$

From (2.8), we have

$$(2.14) \quad u_\beta \nabla_\alpha \dot{r}^{(\alpha\beta)} \equiv \nabla_\alpha [(E^{(\alpha)} + p^{(\alpha)}) u^\alpha + S^\alpha] + u^\alpha u^\beta \nabla_\alpha S_\beta - u^\alpha \partial_\alpha \dot{r}^{(\alpha)} = u^\beta F_\beta^{(\alpha)},$$

$$(2.15) \quad (g^{\alpha\beta} - u^\alpha u^\beta) \nabla_\gamma \dot{r}^{(\alpha\beta)} \equiv (g^{\alpha\beta} - u^\alpha u^\beta) u^\gamma \nabla_\gamma S_\alpha - (g^{\alpha\beta} - u^\alpha u^\beta) \partial_\alpha \dot{r}^{(\beta)} + \\ + [(E^{(\alpha)} + p^{(\alpha)}) u^\alpha + S^\alpha] \nabla_\alpha u^\beta + S^\beta \nabla_\alpha u^\alpha = (g^{\alpha\beta} - u^\alpha u^\beta) I_\alpha^{(\beta)}.$$

Thus, according to (2.9), we obtain for the relativistic equations of radiation field

$$(2.16) \quad u^\alpha \partial_\alpha E^{(\alpha)} + \nabla_\alpha S^\alpha + (E^{(\alpha)} + p^{(\alpha)}) \nabla_\alpha u^\alpha - S_\beta u^\alpha \nabla_\alpha u^\beta = -\sigma'_\alpha (E^{(\alpha)} - a T^\alpha),$$

$$(2.17) \quad u^\alpha \nabla_\alpha S^\beta - (g^{\alpha\beta} - u^\alpha u^\beta) \partial_\alpha \dot{r}^{(\beta)} + (E^{(\alpha)} + p^{(\alpha)}) u^\alpha \nabla_\alpha u^\beta + S^\alpha \nabla_\alpha u^\beta + \\ + S^\beta \nabla_\alpha u^\alpha + u^\alpha u^\beta S_\gamma \nabla_\alpha u^\gamma = -\sigma'_\alpha S^\beta.$$

3. The Characteristics of Relativistic Radiative Hydrodynamics. We study now the Cauchy problem for the system of the relativistic radiative hydrodynamics defined by Einstein equations

$$(3.1) \quad S_{\alpha\beta} = \chi[(rf + E^{(\alpha)} + p^{(\alpha)}) u_\alpha u_\beta - (p + p^{(\alpha)}) g_{\alpha\beta} + u_\alpha S_\beta + u_\beta S_\alpha],$$

the equation of continuity (2.12), the enthalpy equation (2.11), the system of the stream lines (2.13), and the radiation field equations (2.16) and (2.17).

On the hypersurface Σ , with the local equation $x^0 = 0$, we give the values of the potentials $g_{\alpha\beta}$ and of their first derivatives, the values of u_α defining an unitary vector and, for example, the values of the total energy density and the total pressure:

$$(3.2) \quad \begin{cases} rf - p + E^{(\alpha)} = E_t, \\ p + p^{(\alpha)} = p_t. \end{cases}$$

We assume that these data are such that $g_{\alpha\beta} v^{\alpha\beta}$ is normal hyperbolic, that Σ is not tangent to the elementary cones ($g^{00} \neq 0$) and is not tangent to $u(u^0 \neq 0)$. Also, we suppose that the system of local coordinates $\{x^\alpha\}$ is harmonic on Σ , that is, the x^α are local solutions of the Laplace equation

$$F^\alpha = \Delta x^\alpha = g^{\beta\gamma} \Gamma_{\beta\gamma}^\alpha = 0 \quad \text{for } x^0 = 0.$$

We know that we can deduce from the data the values on Σ of the S_α^0 . From (3.2), we have then

$$(3.3) \quad [(rf + E^{(\alpha)} + p^{(\alpha)}) u^0 + S^0] u^\alpha + u^0 S^\alpha = \chi^{-1} S^0_\alpha + (p + p^{(\alpha)}) g^0_\alpha$$

and therefore

$$(3.4) \quad S^\alpha = \frac{1}{u^0} (g^{\alpha\beta} - u^\alpha u^\beta) [\chi^{-1} S^0_\beta + (p + p^{(\alpha)}) g^0_\beta],$$

which define a vector orthogonal to u_α .

We take now the contracted product of (3.3) with u_α , and expressing S^0 by (3.4) for $\alpha = 0$, we get

$$(3.5) \quad rf + E^{(r)} + p^{(r)} = \frac{1}{(u^0)^2} (2u^0 u^\beta - g^{0\beta}) [\chi^{-1} S_\beta^0 + (p + p^{(r)}) g_\beta^0].$$

From here we can deduce the values of $E^{(r)}$ and then of $p^{(r)}$, if

$$(3.6) \quad R = (u^0)^2 + \frac{1}{3} (g^{00} - u^0 u^0) \neq 0.$$

Then, (3.2) give p and rf , and therefore, the values of the thermodynamic variables according to the equation of state. We can consider that we know, thus, the values of f and S .

We note that, according to (2.13), u_α , which is initially unitary, remains unitary.

We assume that the Cauchy data are formal series in the local coordinates, and we also look for series expansions for the solution of the system (3.1), (2.11), (2.12), (2.13), (2.16), (2.17).

Because $g^{00} \neq 0$, we get from (3.1) the values on Σ of the derivatives $\partial_{00} g_{\alpha\beta}$. The equation (2.11) gives, for $u^0 \neq 0$, the value on Σ of $\partial_0 S$. From (2.12) and (2.13) for $\beta = 0$, we have

$$\begin{cases} r\partial_0 u^0 + u^0 r'_f \partial_0 f + \dots = 0, \\ fu^0 \partial_0 u^0 - (g^{00} - u^0 u^0) \partial_0 f + \dots = 0, \end{cases}$$

where the dots denote the terms whose values on Σ are known. From here we obtain the values on Σ of $\partial_0 u^0$ and $\partial_0 f$, if the determinant of system is different from zero, that is

$$(3.6') \quad H = g^{00} - \left(1 - \frac{fr'_f}{r}\right) u^0 u^0 \neq 0.$$

The equation (2.13), for $\beta = i$: $fu^0 \partial_0 u^i + \dots = 0$, gives $\partial_0 u^i$, if $u^0 \neq 0$.

From (2.16) and (2.17) for $\beta = 0$, we have

$$\begin{cases} u^0 \partial_0 E^{(r)} + \partial_0 S^0 + \dots = 0, \\ u^0 \partial_0 S^0 - \frac{1}{3} (g^{00} - u^0 u^0) \partial_0 E^{(r)} + \dots = 0. \end{cases}$$

If $R \neq 0$, with R given by (3.5), we obtain the values on Σ of $\partial_0 E^{(r)}$ and $\partial_0 S^0$, and from (2.17) for $\beta = i$: $u^0 \partial_0 S^i + \dots = 0$, for $u^0 \neq 0$, we get $\partial_0 S^i$.

Thus, under the assumptions

$$(3.7) \quad g^{00} \neq 0, \quad u^0 \neq 0, \quad H \neq 0, \quad R \neq 0,$$

we obtain on Σ the values of the derivatives $\partial_{00} g_{\alpha\beta}$, $\partial_0 f$, $\partial_0 S$, $\partial_0 u^\alpha$, $\partial_0 E^{(r)}$, $\partial_0 S^\alpha$ and the same conditions are valuable for the determination of the succeeding derivatives. If the Cauchy data are analytic, we thus obtain one and only one analytic solution of the system (3.1), (2.11), (2.12), (2.13), (2.16), (2.17).

If $\varphi = 0$ is the local equation of a regular hypersurface, we obtain from (3.7) four types of characteristic hypersurfaces: *gravitational waves*, solutions of equations $g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi=0$; *hypersurfaces generated by the stream lines*, solutions of $u^\alpha\partial_\alpha\varphi=0$; *hydrodynamical waves*, solutions of

$$\left[g^{\alpha\beta} - \left(1 - \frac{fr'}{r} \right) u^\alpha u^\beta \right] \partial_\alpha\varphi\partial_\beta\varphi=0,$$

and *radiative wave fronts*, solutions of

$$\left[u^\alpha u^\beta + \frac{1}{3} (g^{\alpha\beta} - u^\alpha u^\beta) \right] \partial_\alpha\varphi\partial_\beta\varphi=0.$$

4. Theorem of Existence and Uniqueness. We consider as unknown functions $g_{\alpha\beta}$, f , S , u^α , $E^{(n)}$, S^α , satisfying the system defined by the Einstein equations in harmonic coordinates

$$(4.1) \quad R_{\alpha\beta}^{(h)} = \chi \left[(rf + E^{(n)} + p^{(n)}) u_\alpha u_\beta - \left(\frac{1}{2} rf - p + p^{(n)} \right) g_{\alpha\beta} + u_\alpha S_\beta + u_\beta S_\alpha \right],$$

where

$$R_{\alpha\beta}^{(h)} = -\frac{1}{2} g^{\lambda\mu} \partial_{\lambda\mu} g_{\alpha\beta} + f_{\alpha\beta} (g_{\lambda\mu}, \partial_\rho g_{\lambda\mu}),$$

with $f_{\alpha\beta}$ regular given functions, and by the equations (2.11), (2.12), (2.13), (2.16), (2.17).

We shall prove the existence and uniqueness of the solution of the Cauchy problem for this system, using a Leray's theorem for strictly hyperbolic systems [2, §6].

Firstly, we transform the previous system in a Leray system.

The differential system of the stream lines (2.13), by taking the contracted covariant derivative ∇_β , becomes

$$(4.2) \quad \begin{aligned} & (g^{\alpha\beta} - u^\alpha u^\beta) \nabla_\beta \nabla_\alpha f - f u^\alpha \nabla_\beta \nabla_\alpha u^\beta - T g^{\alpha\beta} \nabla_\alpha \nabla_\beta S = \\ & = A(1 \text{ in } g^{\alpha\beta}, 1 \text{ in } f, 1 \text{ in } S, 1 \text{ in } u^\alpha, 1 \text{ in } S^\alpha), \end{aligned}$$

where the notation shows the maximum order of the derivatives appearing in the function A . But, from the equation of continuity (2.12), we have

$$\nabla_\beta u^\beta = -\frac{r'}{r} u^\beta \partial_\beta f + \frac{\sigma_a'}{rf} (E^{(n)} - aT^4)$$

and, according to the Ricci's identity

$$\nabla_\beta (\nabla_\alpha u^\beta) = \nabla_\alpha (\nabla_\beta u^\beta) + R_{\alpha\beta} u^\beta = -\nabla_\alpha \left(\frac{r'}{r} u^\beta \partial_\beta f \right) + R_{\alpha\beta} u^\beta + \nabla_\alpha \left(\frac{\sigma_a'}{rf} (E^{(n)} - aT^4) \right),$$

such that

$$u^\alpha \nabla_\beta \nabla_\alpha u^\beta = -u^\alpha u^\beta \frac{r'}{r} \nabla_\alpha \nabla_\beta f + B(2 \text{ in } g^{\alpha\beta}, 1 \text{ in } f, 1 \text{ in } S, 1 \text{ in } u^\alpha, 1 \text{ in } E^{(n)}).$$

Substituting this into (4.2), we get

$$(4.3) \quad \left[g^{\alpha\beta} - \left(1 - \frac{fr'_f}{r} \right) u^\alpha u^\beta \right] \nabla_\alpha \nabla_\beta f = \\ = T g^{\alpha\beta} \nabla_\alpha \nabla_\beta S + C(2 \text{ in } g^{\alpha\beta}, 1 \text{ in } f, 1 \text{ in } S, 1 \text{ in } u^\alpha, 1 \text{ in } E^{(r)}, 1 \text{ in } S^\alpha).$$

The derivative along the current lines of (4.3) gives

$$(f) \quad u^\gamma \left[g^{\alpha\beta} - \left(1 - \frac{fr'_f}{r} \right) u^\alpha u^\beta \right] \partial_{\alpha\beta\gamma} f = \\ = D(3 \text{ in } g^{\alpha\beta}, 2 \text{ in } f, 2 \text{ in } S, 2 \text{ in } u^\alpha, 2 \text{ in } E^{(r)}, 2 \text{ in } S^\alpha).$$

The equations (2.11) can be rewritten

$$(S) \quad u^\alpha \partial_\alpha S = 0.$$

We apply the differential operator

$$\left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r} \right) u^\lambda u^\mu \right] \nabla_\lambda \nabla_\mu$$

to the system of stream lines (2.13)

$$f u^\alpha \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r} \right) u^\lambda u^\mu \right] \nabla_\lambda \nabla_\mu \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta) \left[g^{\lambda\mu} - \right. \\ \left. - \left(1 - \frac{fr'_f}{r} \right) u^\lambda u^\mu \right] \nabla_\lambda \nabla_\mu \nabla_\alpha f + T g^{\alpha\beta} g^{\lambda\mu} \nabla_\lambda \nabla_\mu \nabla_\alpha S = \\ = E^\beta(3 \text{ in } g^{\alpha\beta}, 2 \text{ in } f, 2 \text{ in } S, 2 \text{ in } u^\alpha, 2 \text{ in } S^\alpha),$$

which, with (4.3) and (2.12), becomes

$$(u^\beta) \quad u^\alpha \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r} \right) u^\lambda u^\mu \right] \partial_{\lambda\mu\gamma} u^\beta = \\ = G^\beta(3 \text{ in } g^{\alpha\beta}, 2 \text{ in } f, 2 \text{ in } S, 2 \text{ in } u^\alpha, 2 \text{ in } E^{(r)}, 2 \text{ in } S^\alpha).$$

Taking into account the equation (2.13), the radiation field equations (2.16) and (2.17) can be written

$$(4.4) \quad u^\alpha \partial_\alpha E^{(r)} + \nabla_\alpha S^\alpha + (E^{(r)} + p^{(r)}) \nabla_\alpha u^\alpha - \\ - S_\beta \left[(g^{\gamma\beta} - u^\gamma u^\beta) \frac{1}{f} \partial_\gamma f - \frac{T}{f} g^{\gamma\beta} \partial_\gamma S + \frac{\sigma'_a}{rf} S^\beta \right] = -\sigma'_a (E^{(r)} - a T^4),$$

$$(4.5) \quad -\frac{1}{3} (g^{\alpha\beta} - u^\alpha u^\beta) \partial_\alpha E^{(r)} + u^\alpha \nabla_\alpha S^\beta + (E^{(r)} + p^{(r)}) u^\alpha \nabla_\alpha u^\beta + \\ + u^\beta S_\gamma \left[(g^{\lambda\gamma} - u^\lambda u^\gamma) \frac{1}{f} \partial_\lambda f - \frac{T}{f} g^{\lambda\gamma} \partial_\lambda S + \frac{\sigma'_a}{rf} S^\gamma \right] + S^\beta \nabla_\alpha u^\alpha = -\sigma'_a S^\beta.$$

Now we take the derivative of (4.4) along the current lines, the contracted covariant derivative of (4.5) and we subtract the two equations. We obtain

$$(4.6) \quad [u^\alpha u^\beta + \frac{1}{2}(g^{\alpha\beta} - u^\alpha u^\beta)] \nabla_\alpha \nabla_\beta E^{(r)} - \frac{2}{f} \left(1 - \frac{fr'_f}{r}\right) S^\alpha u^\beta \nabla_\alpha \nabla_\beta f = \\ = H(2 \text{ in } g^{\alpha\beta}, 1 \text{ in } f, 1 \text{ in } S, 1 \text{ in } u^\alpha, 1 \text{ in } E^{(r)}, 1 \text{ in } S^\alpha).$$

We apply then the operator

$$\left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r}\right) u^\lambda u^\mu \right] \nabla_\lambda \nabla_\mu$$

to the equation (4.6) and we take into account equation (f). We get

$$(E^{(r)}) \quad \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r}\right) u^\lambda u^\mu \right] \left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta) \right] \partial_{\alpha\beta\lambda\mu} E^{(r)} = \\ = I(4 \text{ in } g^{\alpha\beta}, 3 \text{ in } f, 3 \text{ in } S, 3 \text{ in } u^\alpha, 3 \text{ in } E^{(r)}, 3 \text{ in } S^\alpha).$$

The operator

$$u^\rho \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r}\right) u^\lambda u^\mu \right] \left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta) \right] \nabla_\alpha \nabla_\beta \nabla_\lambda \nabla_\mu \nabla_\rho$$

applied to the equation (2.7), taking into account the equations (2.11), (E^(r)), (f), (u^β), yields

$$(S^r) \quad u^\rho \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r}\right) u^\lambda u^\mu \right] \left[u^\alpha u^\beta + \frac{1}{3}[(g^{\alpha\beta} - u^\alpha u^\beta)] \partial_{\alpha\beta\lambda\mu\delta\rho} S^r = \\ = J^r(6 \text{ in } g^{\alpha\beta}, 5 \text{ in } f, 5 \text{ in } S, 5 \text{ in } u^\alpha, 5 \text{ in } E^{(r)}, 5 \text{ in } S^\alpha).$$

Finally, the derivative along the stream lines of the Einstein equations (4.1) gives

$$(g^{\alpha\beta}) \quad -\frac{1}{2} u^\gamma g^{\lambda\mu} \partial_{\lambda\mu\gamma} g^{\alpha\beta} = K^{\alpha\beta}(2 \text{ in } g^{\alpha\beta}, 1 \text{ in } f, 1 \text{ in } S, 0 \text{ in } u^\alpha, 1 \text{ in } E^{(r)}, 1 \text{ in } S^\alpha).$$

The system defined by the equations (g), (f), (S), (u^β), (E^(r)), (S)^r is a hyperbolic Leray system. The matrix of this system, of type 21×21, has for diagonal elements

$$a(g^{\alpha\beta}) = -\frac{1}{2} u^\gamma g^{\lambda\mu} \partial_{\lambda\mu\gamma}, \quad a(f) = u^\gamma \left[g^{\alpha\beta} - \left(1 - \frac{fr'_f}{r}\right) u^\alpha u^\beta \right] \partial_{\alpha\beta\gamma}, \quad a(S) = u^\alpha \partial_\alpha, \\ a(E^{(r)}) = \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r}\right) u^\lambda u^\mu \right] \left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta) \right] \partial_{\alpha\beta\lambda\mu}, \\ a(S^r) = u^\delta u^\rho \left[g^{\lambda\mu} - \left(1 - \frac{fr'_f}{r}\right) u^\lambda u^\mu \right] \left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta) \right] \partial_{\alpha\beta\lambda\mu\delta\rho}.$$

This system satisfies the assumptions on the derivatives with the following indices for the unknowns:

$$s(g^{\alpha\beta})=7, \quad s(f)=s(S)=s(u^\beta)=s(E^{(r)})=s(S^r)=6;$$

the following indices for the equations:

$$t(g^{\alpha\beta})=5, \quad t(f)=4, \quad t(S)=6, \quad t(u^\beta)=4, \quad t(E^{(r)})=3, \quad t(S^r)=1;$$

and therefore

$$m(g^{\alpha\beta})=3, \quad m(f)=3, \quad m(S)=1, \quad m(u^\beta)=3, \quad m(E^{(r)})=4, \quad m(S^r)=6.$$

To prove this it is sufficient to observe that the maximum orders of the derivatives of the unknown functions, which can appear in the coefficients and in the right-hand members of the system, corresponding to these indices, are

	$g^{\alpha\beta}$	f	S	u^β	$E^{(r)}$	S^r
$(g^{\alpha\beta})$	2	1	1	1	1	1
(f)	3	2	2	2	2	2
(S)	1	0	0	0	0	0
(u^β)	3	2	2	2	2	2
$(E^{(r)})$		3	3	3	3	3
(S^r)	6	5	5	5	5	5

These maximum orders are compatible with our system.

On the hypersurface Σ , with the local equation $x^0 = 0$, we give as initial data the values of the gravitational potentials $g^{\alpha\beta}$ and of their first derivatives, the values of u^β defining an unitary vector, the values of the total pressure $p + p^{(r)}$ and of the total energy density $rf - p + E^{(r)}$. We suppose that these values satisfy the assumptions: a) the quadratic form $g_{\alpha\beta}v^\alpha v^\beta$ is normal hyperbolic; hypersurface Σ is space-like in each point with respect to the corresponding elementary cone; b) we have on $\Sigma: F^p = 0$; c) the relations $S_\alpha^0 = \chi T_\alpha^0$ and the initial data give admissible values for $f, S, E^{(r)}, S^r$ with $f r_f \geq r$.

The equations (4.1) give the values on Σ of the second derivatives of $g^{\alpha\beta}$; the equation (S) gives the value of the first derivative of S . The equations (2.12) and (2.13) give, as we know, the values of the derivatives of f and u^β and f and u^β , and (2.16) and (2.17) the values of the derivatives of $E^{(r)}$ and S^r . The equations $(g^{\alpha\beta})$ give the third derivatives of $g^{\alpha\beta}$, and (S), by differentiation, gives the second derivative of S . The equation (4.3) gives then the second derivative of f ; by differentiation, (2.13) gives the second derivative of u^β ; (4.6) gives the second derivative of $E^{(r)}$, and by differentiation, (2.17) the second derivative of S^r . The equations $(g^{\alpha\beta})$, (f), (S), (u^β) , (4.6), $(E^{(r)})$, (2.17) and their succeeding derivatives give the derivatives of $g^{\alpha\beta}$ of order ≤ 6 , and of $f, S, u^\beta, E^{(r)}, S^r$ of orders ≤ 5 . Thus we have obtained the values on Σ of the derivatives of orders $\leq s(\sigma) - 1$.

Thus we get a Cauchy problem for the system $(g^{\alpha\beta}) - (S^r)$ which is a strictly hyperbolic system. From the Leray's theorem we obtain

Theorem 5.1. *If the initial data are sufficiently regular, the Cauchy problem for the system $(g^{\alpha\beta}), (f), (S), (u^\beta), (E^{(r)}), (S^r)$ has one and only one solution $(g^{\alpha\beta}, f, S, u^\beta, E^{(r)}, S^r)$.*

An argument similar to that of [4] shows that this solution is a solution of the system (3.1), (2.11), (2.12), (2.13), (2.16), (2.17).

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TEOREMĂ DE EXISTENȚĂ ȘI UNICITATE ÎN HIDRODINAMICA RADIATIVĂ RELATIVISTĂ

(Rezumat)

Se dă o teoremă de existență și unicitate a soluției problemei lui Cauchy pentru ecuațiile hidrodinamicii radiative relativiste, extinzând teoremele obținute de Fourès-Bruhat și Lichnerowicz, la cazul când mișcarea fluidului se face în prezența unui câmp de radiație.

Theorem 2.1. If the initial data are sufficiently regular, the Cauchy problem for the system (6₁), (1), (2), (4₁), (6₂), (5₁) has one and only one solution (6₁), (2), (4₁), (6₂), (5₁).

An argument similar to that of (4) shows that this solution is a solution of the system (3.1), (2.1), (2.1), (2.1), (2.1), (2.1).

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TEOREMA DE EXISTENȚĂ ȘI UNICITATE ÎN HIDRODINAMICĂ RADIATIVĂ RELATIVISTĂ

(Rezumat)

Se dă o teoremă de existență și unicitate a soluției problemei lui Cauchy pentru ecuațiile hidrodinamice relativiste, extinzând teoremele obținute de Fowler-Bryant și Lichnerowicz, în cazul unui mijloc relativist de fluid și în prezența unui câmp de radiație.

CONTRIBUTION À L'ÉTUDE DU MÉCANISME DE L'EFFET PROCOPIU

PAR

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L'effet Procopiu [1] est un phénomène d'aimantation d'une substance ferromagnétique solide à la symétrie cylindrique, sous l'action d'un champ magnétique circulaire alternatif et d'un champ magnétique longitudinal continu. L'aimantation est mise en évidence par la f.e.m. induite dans une bobine coaxiale à l'échantillon cylindrique.

Ce genre d'aimantation a été nommé par Hofbauer et Koch [2] *effet Procopiu*.

Dans la littérature de spécialité on a cité plusieurs essais d'expliquer le mécanisme de l'effet Procopiu en utilisant les résultats expérimentaux obtenus sur les substances ferromagnétiques solides sous forme de fils [3], [4] ou de couches déposées sur un fil nonmagnétique [5].

Dans l'ouvrage ci-présent on donne quelques résultats sur l'effet Procopiu en utilisant les échantillons des colloïdes stables surnommés aussi ferrofluides [6]. On fait une comparaison théorique et expérimentale avec le cas de matériaux solides.

Pour les expériences on a utilisé une installation ressemblante à celle utilisées à l'étude des corps solides [7].

La fig. 1 reproduit dans la partie inférieure la dépendance de la f.e.m. de type Procopiu E_p et dans la partie supérieure le champ magnétique circulaire en fonction du temps pour un colloïde de Fe_3O_4 préparé à base du pétrole ayant une concentration volumétrique de $\varepsilon = 0,04$. Tout comme dans le cas des corps solides [3] la fréquence de la f.e.m. E_p pour les colloïdes est le double de la fréquence du champ magnétique alternatif circulaire h_a . Mais, par rapport aux substances solides où la forme de la courbe est tangentielle pour un champ magnétique longitudinal h_1 faible et sinusoïdale pour des champs h_1 forts, pour les colloïdes elle est seulement sinusoïdale.

Dans une aimantation du type Procopiu, d'une colloïde magnétique, où la $h_a = h_{am} \sin \omega t$ a la fréquence de 50 Hz, le vecteur d'aimantation $\vec{M}$ est colinéaire avec $\vec{H}$ parce qu'un ferrofluide est parfaitement mou du point de vue magnétique, l'aimantation étant faite dans un intervalle plus court de $1\mu\text{s}$ [8]. De la fig. 2 on observe que

$$(1) \quad M_1 = M \sin \varphi = M \frac{h_1}{H},$$

où l'aimantation M d'un ferrofluide au cas d'une distribution uniforme des dimensions des particules est décrite par l'équation

$$(2) \quad M = \varepsilon M_s \mathcal{L} \left(\frac{\mu_0 V M_s H}{kT} \right),$$

où $\mathcal{L}$ représente la fonction de Langevin, M_s — l'intensité d'aimantation des particules solides, ε — la concentration volumétrique du solide magnétique, V — le volume d'une particule magnétique, μ_0 — la perméabilité de l'air, k — la constante du Boltzman et T — la température.

Suivant le raisonnement du Skorski et Duracz [3] pour calculer la f.e.m. du type Procopiu pour les matériaux solides nous avons obtenu pour les colloïdes magnétiques la relation suivante :

$$(3) \quad E_p(t, h_1, h_m) = \frac{1}{2} N S_f \varepsilon \omega h_1 h_{am}^2 \sin^2 \omega t \left[\frac{\mu_0 M_1}{(h_1^2 + h_a^2)^{3/2}} - \frac{2kT}{V} \frac{1}{(h_1^2 + h_a^2)^2} \right],$$

avec la condition

$$(4) \quad H \geq \frac{kT}{\mu_0 M_s V},$$

où N représente le nombre des spires de la bobine sonde et S_f — la section de la boucle de colloïde magnétique.

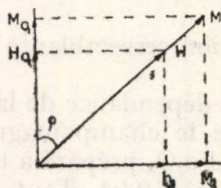


Fig. 2 — L'aimantation d'un grain monodomainique en l'aimantation de type Procopiu.

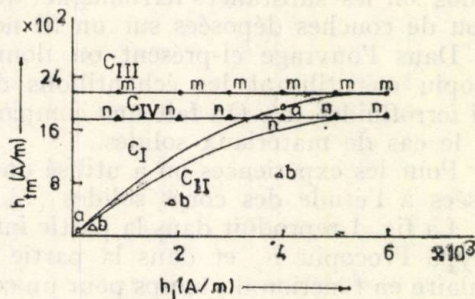


Fig. 3 — La comparaison des résultats expérimentaux aux résultats théoriques.

Si $h_1 > h_{am}$, c'est-à-dire le h_{am} est négligé par rapport au h_1 , suivant le même raisonnement comme dans le cas d'une épreuve solide [5] on obtient pour le champ magnétique alternatif h_{tm} (qui est égal à $h_{am} \sin \omega t$), où la f.e.m. $E(t, h_1, h_{am})$ présente la valeur maxima, la relation

$$(5) \quad h_{tm} = \frac{h_{am}}{\sqrt{2}}.$$

La relation (5) est différente de l'expression du h_{tm} établie pour les matériaux magnétiques solides [5] où $h_{tm} = f(h_{am}, h_1)$.

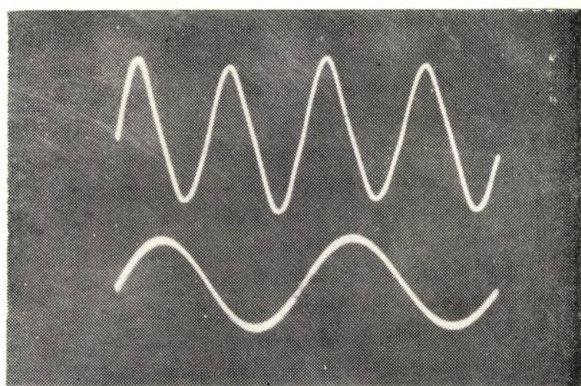


Fig. 1— La dépendance de la f.e.m. de type Procopiu E_p (en dessous) et du champ magnétique circulaire (en dessus) en fonction du temps.

Dans la fig. 3 on donne les courbes théoriques calculées selon [5] pour les épreuves solides (C_I et C_{II}), celles calculées d'après la relation (5) pour les colloïdes (C_{III} et C_{IV}) avec les mêmes valeurs du champ alternatif $h_{am} = 3\,200\text{ A/m}$ (C_I , C_{III}) et $h_{am} = 2\,880\text{ A/m}$ (C_{II} , C_{IV}).

Les points a sont les valeurs expérimentales données dans [3], pour des fils de permalloy aimantés à la saturation.

Les points b sont les valeurs expérimentales d'après [9] pour des fils de nickel aimantés au dessous de la saturation.

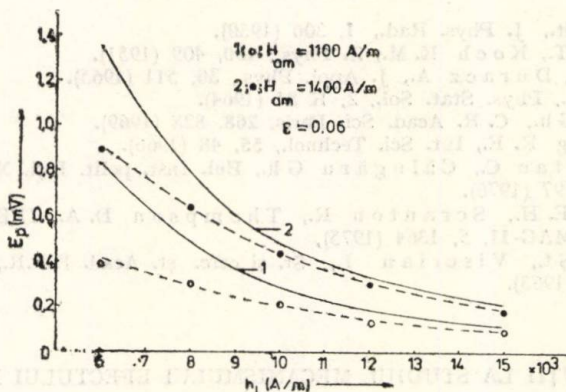


Fig. 4 — La variation de E_p en fonction de h_1 .

Les points m et n sont les valeurs expérimentales obtenus pour un colloïde avec des particules monodomaines (la dimension moyenne $150-200\text{Å}$) avec une concentration volumétrique $\epsilon = 0,04$ (on a obtenu aussi des valeurs identiques pour les colloïdes avec $\epsilon = 0,01$; $\epsilon = 0,06$).

La concordance parfaite des données expérimentales avec celles théoriques au cas des colloïdes est du fait que les particules monodomaines d'une colloïde stabilisé n'actionnent pas entre elles. On explique de la même manière la forme sinusoïdale du E_p quelles que soient les valeurs de h_1 .

Dans la fig. 4 les courbes continues représentent les courbes théoriques $E_p = f(h_1)$ calculées d'après la relation

$$(6) \quad E_p(t_m, h_1, H_{am}) = \frac{1}{2} N S_f \epsilon \omega H_{am}^2 \left(\frac{\mu_0 M_s}{h_1^2} - \frac{2kT}{V} \frac{1}{h_1^3} \right),$$

obtenue pour $t = t_m$, pour colloïde magnétique avec $\epsilon = 0,06$, pour deux valeurs du champ h_{am} .

On observe un rapprochement des courbes expérimentales et théoriques pour les valeurs élevées de h_1 . Cela grace au fait que la f.e.m. E_p donnée par la relation (3) a été calculé dans la condition (4) satisfaite pour des valeurs élevées de h_1 .

La comparaison des résultats expérimentaux et théoriques permet de conclure :

a) la relation (3) exprime correctement la valeur de E_p quand la condition (4) est satisfaite, ce qu'on peut réaliser pour des champs h_1 forts ;

b) la relation (5) exprime correctement la valeur du champ magnétique pour lequel la force électromotrice $E(t, h_1)$ possède des valeurs maxima $E(t_m, h_1)$ au cas des colloïdes magnétiques, quelle que soit la concentration volumétrique ε .

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Chaire de Physique

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CONTRIBUȚII LA STUDIUL MECANISMULUI EFECTULUI PROCOPIU

(Rezumat)

Se studiază forma efectului Procopiu la coloizii magnetici stabili; se compară atît pe cale teoretică cît și experimentală comportarea lor cu cea a materialelor solide.

$$E(t, h_1) = \frac{1}{2} \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0} \right) \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0} \right) \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0} \right) \quad (6)$$

STUDII DE MICROSCOPIE OPTICĂ PRIVIND IDENTIFICAREA DEFECTELOR DIN FIRELE POLIESTERICE ȘI STABILIREA NATURII LOR

DE

SOFIA MELINTE, MARCEL MOISE și N. MATEESCU

1. Introducere. Una din cauzele principale ce duce la micșorarea calității firelor sintetice este formarea pilingului precum și urmările ce decurg prin prelucrarea lor ulterioară. Premisa pentru înlăturarea acestor deficiențe de către producător este cunoașterea exactă a legăturilor dintre spectrul lor de apariție și parametrii producției.

Apariția pilingului, se consideră în primul rînd că s-ar datora deficiențelor de filare, care în mare măsură se datorează neomogenităților topiturii, neomogenități care vor duce, în timpul procesului de filare, la apariția îngroșărilor locale în fibrele elementare.

Pot apărea însă îngroșări în fibrele singulare și în cazul cînd acestea parcurg după filare un proces de tratament termic necorespunzător, în timpul etirării. Acest tip de defecte se numesc defecte textile.

Defectele apărute în timpul celor două procese amintite duc ulterior, în timpul prelucrării firelor, la ruperi ale fibrelor elementare și în cazul suprafețelor textile, la micșorarea calității acestora. Pe baza cercetărilor efectuate de noi și de alți autori [1] — [4] s-a considerat utilă împărțirea defectelor în trei tipuri: cu incluziuni, fără incluziuni și diverse.

În cercetările noastre ne-am limitat în special la metodele de punere în evidență a defectelor, identificarea lor, precum și la stabilirea naturii și cauzei care le-a produs și a influenței pe care o au asupra produsului finit.

2. Defecte cu incluziuni. 2.1. Incluziuni de TiO_2 . În urma cercetării fibrelor elementare pe o lungime apreciabilă, în lumină polarizată, cu un microscop de tip Pol. U, s-a constatat apariția frecventă a unor îngroșări locale ale fibrelor, care pot fi centrale sau superficiale. Prin încercări de solvire sub microscop, în acid sulfuric concentrat (cca 80%), s-a constatat că solvirea începe de preferință la periferia defectului, care datorită stării mai reduse de aranjare structurală este mult mai accesibilă mijlocului de umflare. După solvirea completă a mantalei ce înconjoară incluziunea, pe suportul obiectivului rămîne o incluziune compactă, care prin apăsare pe placa de acoperire se desface într-un număr mare de particule; s-a constatat că ele au dimensiunile normale ale pigmentilor de matisare pentru firele elementare de polietilentereftalat (PET) și nu sînt altceva decît particule de TiO_2 .

Diversele forme de apariție a acestor incluziuni în fibrele elementare de PET sînt redată în fig. 1 (*a, b, c, d*) din care se observă că sînt centrale, aproximativ sferice, alungite și care în procesul de filare-etirare ajung la suprafața fibrei. Nu rare ori sînt cazurile cînd incluziuni mai mari de TiO_2 produc rupele ale fibrelor elementare în procesul de etirare sau de utilizare a acestora în industria textilă, ceea ce se reflectă prin apariția unor aglomerări de fibre în nodurile produselor textile. Apariția particulelor mai mari de TiO_2 , în topitura de polimer din care se trag fibrele poliesterice, în faza actuală a tehnicii folosite, nu este cauzată decît în mică măsură de o mărunțire insuficientă și dispersare a pigmentilor de matisare, ci credem că se datorește mai mult, așa cum au constatat și alți autori [1], [2], aglomerările ulterioare, în procesul de transesterificare și policondensare. Aglomerările formate în cele două procese se vor păstra în întregul proces de producție, micșorînd calitatea firelor, implicit a produselor textile.

2.2. Incluziuni datorate substanței de polimer deteriorat. În cazul filării din topitură, datorită reacțiilor de disociere ce au loc în timpul procesului de topire a polimerului, apoi datorită gradientilor de temperatură relativ ridicați și menținuți un timp îndelungat în topitură, a supraîncălzirii locale la pereții dispozitivelor, respectiv la colțurile moarte ale capului de filare etc., se produc degradări termice ale substanței polimerice din care se extrag firele. În cazul procedeului de filare cu grătar, asemenea deteriorări sînt vizibile, evidențiate de formarea craterelor de filare, cînd interiorul capului de filare nu este curățat un timp mai îndelungat de funcționare.

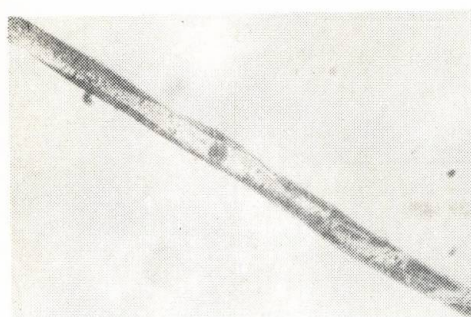
În cazul cînd particulele din topitura poliesterică, datorate cauzelor amintite, nu sînt reținute de filtrul duză, ele ajung în firul elementar filat, care după etirare duce la îngroșări vizibile ale firelor obținute.

În urma unei analize microscopice prin transparentă, prin reflexie sau prin nuanțare (colorare) s-a putut pune în evidență acest tip de defecte. Forma lor de apariție este redată în fig. 2 (*a, b, c, d*).

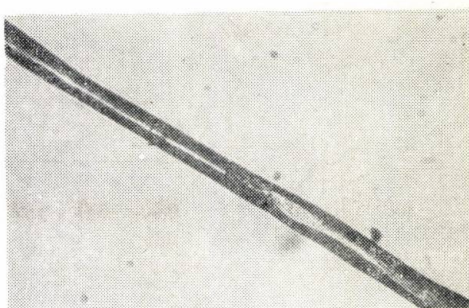
Incluziuni mai mari, în cazul acestui tip de îngroșări, conduc la ruperea firelor în timpul procesului de etirare sau în cel de prelucrare ulterioară în industria textilă. În cazul cînd nu se rupe firul în timpul etirării, datorită subțierilor înainte și după îngroșare, apare pilingul, sau răsuciri locale ale firelor elementare care duc ulterior la ruperea firului în procesul urzitelui etc.

Pe lîngă cele două tipuri de defecte vizibile prin reflexie sau transmisie, în cazul analizelor microscopice efectuate de noi în lumină polarizată, s-au identificat și incluziuni care erau puse în evidență de electrometru și erau invizibile la microscop. Prin încercări de umflare a acestor fire în acid sulfuric de diverse concentrații s-a constatat apariția unor defecte care datorită stării lor structurale, diferită de a restului firului, sînt mai rezistente la solvire, dimensiunile lor nedepășind $5 \dots 6 \mu\text{m}$. Pentru a stabili natura lor s-au izolat îngroșări locale ale firelor elementare filate, care s-au solvit încet în acid sulfuric. Credem că apariția lor în firul filat se datorește deteriorării topiturii produsă de cauzele amintite și care mai avînd încă o vîscozitate suficientă și la ieșirea din duză ia forma unei dungi alungite, mai puțin vizibile în firele albe dar foarte vizibile în firele colorate, unde produce defecte.

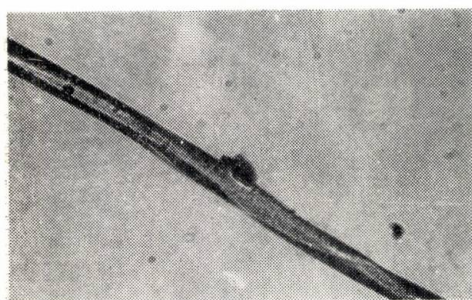
Pentru a stabili structura acestui defect s-au efectuat numeroase măsurători de birefringență în diferite locuri ale umflăturii, constatîndu-se



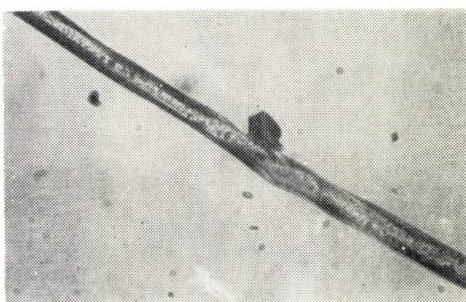
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b

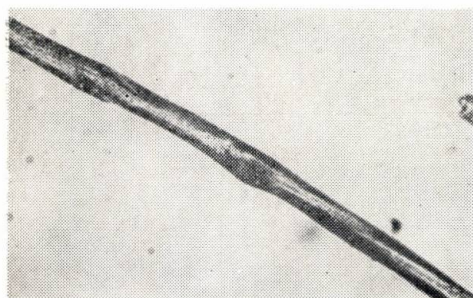


c

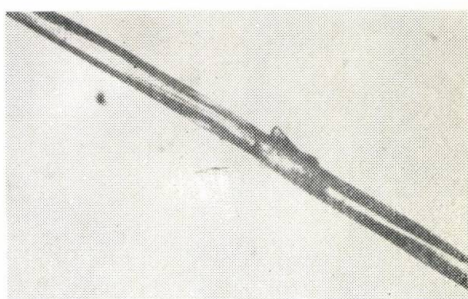


d

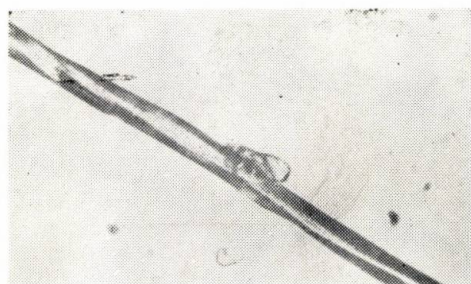
Fig. 1



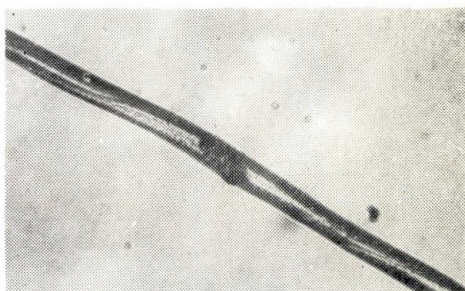
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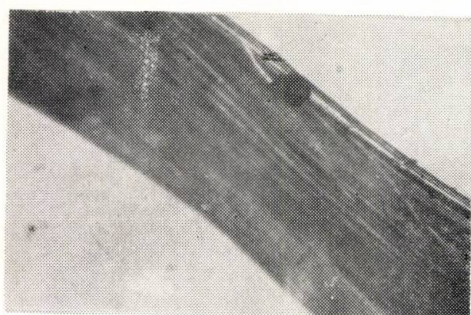


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Fig. 2



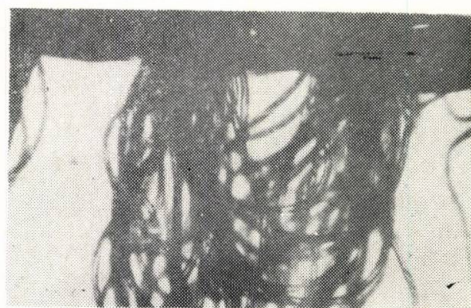
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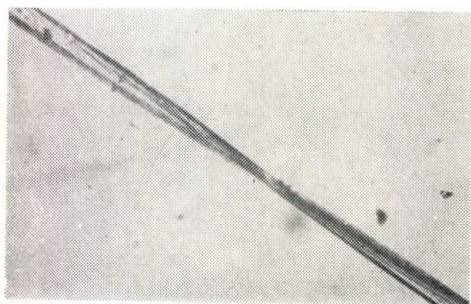


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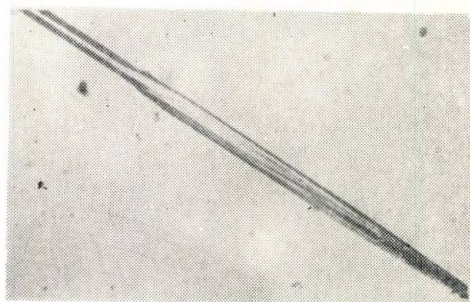


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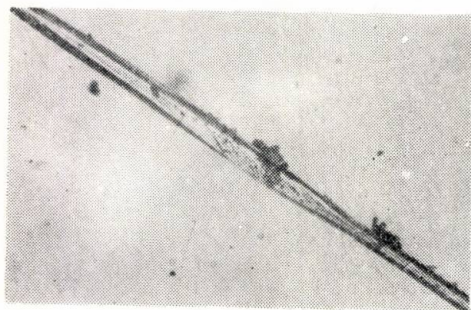
Fig. 3



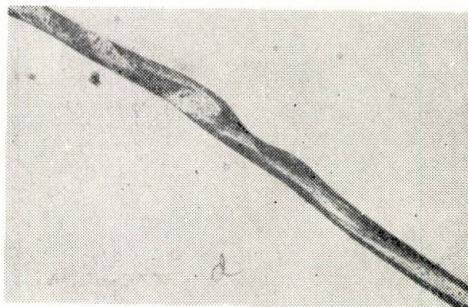
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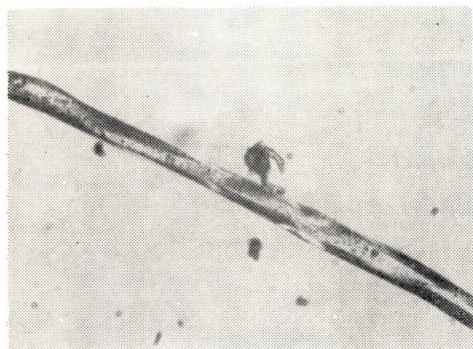


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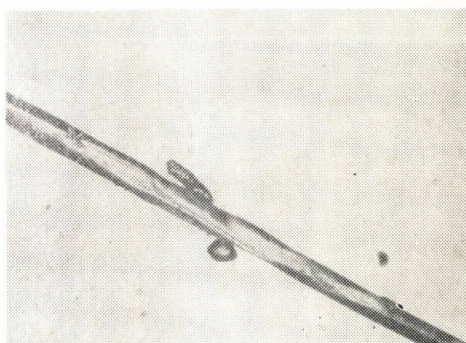


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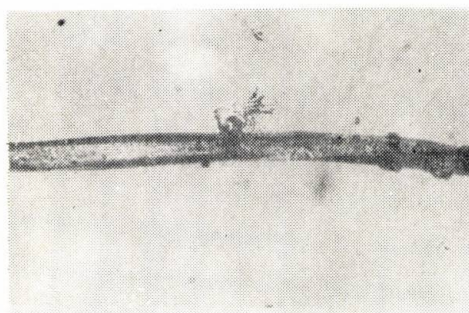
Fig. 4



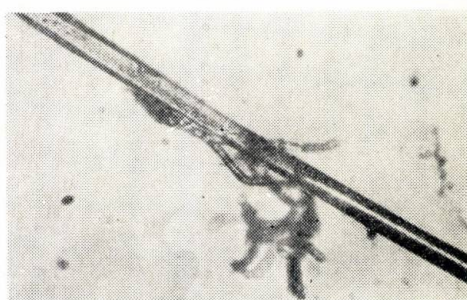
a



b



c



d

Fig. 5



o scădere rapidă a birefringenței față de firele normale; pentru centrul ei s-a găsit o valoare de $4,3 \cdot 10^{-3}$, corespunzătoare firului filat.

Tot prin microscopie optică s-a pus în evidență în această lucrare, existența în firele poliesterice a unor defecte de înaltă cristalinitate, care se topesc la o temperatură mai mare decât restul firului; ca mărime, ele se situează sub 0,1 mm în diametru și au 0,2 ... 0,4 mm în lungime. Frecvența lor de apariție fiind foarte redusă, li s-a dat mai puțină atenție.

De asemenea, o frecvență de apariție redusă o au defectele ce conțin incluziuni mecanice, granule de nisip etc.

3. Concluzii privind influența defectelor asupra produsului finit.

Analizând sub microscop în contrast de fază un defect dintr-un produs finit, se constată prezența diverselor defecte amintite sub diverse forme (rotunde, alungite, capete de filamente rupte, aglomerări ale acestor filamente în nodurile țesăturii etc), care apar ca defecte evidente în suprafața produsului finit, vizibile în fig. 3 (a, b, c, d).

Datorită diverselor incluziuni, în anumite porțiuni firele se vor etira mai puțin, ducând în final la îngroșări. Forțele de tensiune ce apar în timpul etirării pot duce la ruperea unor fire elementare și apoi la aglomerarea lor în timpul prelucrării ulterioare, fapt evidențiat în fig. 3, sau la variații de grosime, înainte și după apariția acestora, fenomen evidențiat de microfotografiile din fig. 4 (a, b, c, d). În cazul solicitării lor la întindere în timpul prelucrării ulterioare, se vor produce ruperi foarte dese și deci formare de piling.

Este cunoscut faptul că filamentele ce pleacă de la filieră trec peste un panou de preparație (o emulsie apoasă de uleiuri, emulgatori, agenți de înmuiere și antistatizanti în proporții bine stabilite) și sînt apoi înmănunchiate în timpul procesului de filare sub forma unui cablu (fir) format din mai multe filamente. Preparația are rolul de a conferi firului filat o prefinisare.

În timpul cercetărilor noastre privind procesul de filare-etirare am constatat că filamentele de PET, ce vin de la cele două capete de filare (16×2), după trecerea peste panoul de preparație, devin unele centrale și altele laterale. Filamentele centrale se încarcă mai puțin cu preparație, în timp ce cele laterale mai mult, întrucît în partea centrală a panoului preparația este mai fluidă, iar pe părțile laterale mai vîscoasă, fapt ce duce la antrenarea sa de către fire. În procesul de etirare și termofixare această preparație se solidifică pe fire dînd neuniformități de grosime, evidențiate de microfotografiile din fig. 5 (a, b, c, d).

De asemenea, preparația de pe firele prea încărcate, are ca urmare depunerea ei pe diversele părți ale mașinilor cu care se prelucreează firele, micșorînd orificiul de trecere al firului; prin frecare se produc ruperi de fire singulare și adunarea lor sub formă de defecte, ceea ce micșorează calitatea produsului finit.

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OPTICAL MICROSCOPY STUDIES ON IDENTIFICATION OF POLYESTER FIBRES DEFECTS AND ESTABLISHMENT OF THEIR NATURE

(Summary)

Various techniques of optical microscopy by using transmission and reflection of polarised light are used to study polyester fibres, especially of polyethyleneterephthalate (PET) type. The types and nature of defects are found, as well as the causes which determine defects in finished products.

EQUATION OF DEBYE TYPE FOR THERMOSTIMULATED PROCESSES IN STRONG ELECTRIC FIELDS

BY

RODICA NEAGU

1. Introduction. Thermally stimulated measurements constitute a recent technique [1] of investigation of dielectric loss mechanisms, of the trapping and recombination levels due to structural imperfections (impurities and dopants), of the influence of molecular structure on charge storage and conduction processes in dielectrics, electrets and ferro-electrics, etc.

In this work we study the dielectric non isothermal relaxation in polymers, which is due to the reorientation of the permanent dipoles. This phenomenon is obtained by a field-temperature treatment.

The dielectrics bearing a persistent charge are called electrets.

In the following we intend to clarify from a theoretical point of view, some of the phenomena which contribute to the discharge of electrets. These phenomena are due to the reorientation of the existing permanent dipoles.

After the switching of the external electric field, the electret does not discharge immediately. Only that part of the polarization obtained by electronic polarization of the molecules, disappears instantaneously. The polarization due to the dipoles orientation modifies with temperature, because the internal frictional forces depend exponentially on temperature. By heating the decay of polarization in time is accelerated.

The electrets create a strong external electric field of about 30 kV/cm which can be utilized for various technical purposes. This is the reason why the fundamental study of electret phenomenon is necessary. The recentest theoretical studies [2]—[5] are based on the thermostimulated discharge (TSD) of the electret, i.e. the theoretical study of the phenomena which cooperate when the electret comes back to the neutral state by means of thermostimulated processes. The study of this phenomena led to a better understanding of the forming and discharging mechanisms of electrets.

The charge of the electret is mainly constituted by the dipolar orientation, the existence of a superficial charge and by the existence of spatial charge. The thermostimulated discharge of the electret will give rise in the external circuit to a current which will have several peaks. These peaks correspond to different discharge mechanisms, which can be studied with a great accuracy by TSD technique.

The forming conditions of the thermoelectrets have a great importance on the modification of the contribution of different mechanisms of charge decay.

There are three forming parameters which characterise the formation of the thermoelectrets: temperature T_f , time t_f and applied electrical field E_f . This accounts for the importance of the study of the forming process of the thermoelectret, i. e. to obtain an equation which gives the change of the polarization during the formation of the thermoelectret and the value of the polarization at the end of the charging period as a function of the three characteristic parameters T_f , t_f , E_f . This value of the polarization is the greatest obtained during a TSD cycle and constitutes the initial polarization for the thermostimulated discharge of the electret.

The equation obtained by Debye [6] for the variation in time of the dielectric displacement D , of a dielectric acted by an external field E , is

$$(1) \quad \tau(1-p_0) \frac{dD}{dt} + (1-p_0-p_1)D = \tau(1+2p_0) \frac{dE}{dt} + (1+2p_0+2p_1)E,$$

where $D(t)$ is the dielectric displacement; t is the time; τ is the relaxation time; $\alpha=1/\tau$ is the relaxation frequency; $p_0=(4\pi/3)n\alpha_0$; $p_1=(4\pi/3)n\alpha_1$; α_0 is orientational or electronic polarizability; α_1 is the dipolar polarizability; $E(t)$ is the applied electric field.

Debye deduces this equation in the following hypothesis: a) the temperature remains constant; b) the applied electric field is weak; c) the internal Lorentz field is $F=E+4\pi P/3$, so the interaction of the molecular dipoles of the dielectric is neglected.

2. The Polarization Equation for Polar Polymers characterized by a Single Value of the Relaxation Time in Weak Fields. The extension of the Debye equation to deduce the variation in time of the polarization P for polymers can be done in following hypothesis.

i) We suppose that the polymer does not contain charge, so, the internal electric field and the dipolar polarization are independent of the spatial coordinates (uniform in whole of the polymer). In fact, the dipolar polarization is often accompanied by a polarization produced by a spatial charge of the electret. In this way the internal electric field and the polarization become non-uniform, and they depend on the spatial coordinates [7]. In this case the theory remains true if we consider the internal electric field and the polarization to be averaged over the thickness of the polymer.

ii) In the equation (1) τ is the relaxation time and $\alpha=1/\tau$ is the relaxation frequency. In this theory we suppose that $\tau(T)$ (function of temperature [7]).

iii) For the polymers we can suppose with a good approximation that [7]

$$(2) \quad (\epsilon_s - \epsilon_\infty)_T = \frac{n\mu^2}{3\epsilon_0 K T},$$

where ϵ_s is the static dielectric constant; ϵ_∞ —the optical dielectric constant; n — the number of permanent dipoles reported to the unity volume; μ — the dipole moment; K — Boltzmann's constant; ϵ_0 — the permittivity of vacuum.

We replace in Eq. (1) $D=E+4\pi P$ and obtain

$$(3) \quad \frac{dP}{dt} + \frac{1-p_0-p_1}{\tau(1-p_0)} P = \frac{3p_0}{4\pi(1-p_0)} \frac{dE}{dt} + \frac{3(p_0+p_1)}{4\pi\tau(1-p_0)} E.$$

Let us deduce from Eq. (3) the equation for the orientational polarization (i.e. slow polarization). We can impose the condition $p_0=0$ because $\alpha_0=0$ and get

$$\frac{dP_s}{dt} + (1-p_1)\alpha(T)P_s = \frac{3p_1}{4\pi} \alpha(T)E;$$

but

$$p_1 = \frac{4\pi}{3} n \frac{\mu^2}{3KT}.$$

Because $p_1 \ll 1$ we finally obtain

$$(4) \quad \frac{dP_s}{dt} + \alpha(T)P_s = \frac{n\mu^2}{3KT} \alpha(T)E.$$

This equation does not depend on the chosen system of unities. If we make substitution (2), we get in SI system

$$(5) \quad \frac{dP_s}{dt} + \alpha(T)P_s = \epsilon_0(\epsilon_s - \epsilon_\infty)_T \alpha(T)E.$$

In the case of a short-circuit ($E=0$) we get

$$(5') \quad \frac{dP_s}{dt} + \alpha(T)P_s = 0.$$

This is the equation used by Turnhout in TSD theory for the explanation of peaks obtained due to the dipole reorientation [7].

3. The Polarization Equation for Polar Polymers characterized by a Single Value of the Relaxation Time in Strong Fields. Generally, the polarization P depends on the intensity of the applied electric field. In the case of dielectrics, only small displacements of the charges occur. The reactive forces that appear in this way are proportional to the displacements, but not for great electric field's intensities.

In addition the applied electric fields tends to orientate the permanent dipoles. In both cases the applied electric field generates a dipolar density:

the electric field polarizes the dielectric, i.e. $\vec{P} = \chi \vec{E}$.

This simple relation is valid only for moderated intensities of the field. At higher than 10^4V/cm intensities of the field the deviations from this relation become important. The electrets obtained by us [9], [10] or by others [7] have been prepared in fields higher than 10^4V/cm . In this case we must consider [10] $\vec{P} = \chi \vec{E} + (\xi E^2) \vec{E}$.

The series development of $\vec{P}$ contains only odd powers of $\vec{E}$, because a reversal of the direction of $\vec{E}$ leads to a reversal of the direction of $\vec{P}$: $\vec{P}(\vec{E}) = -\vec{P}(-\vec{E})$.

In the case of normal saturation, obtained in strong electric fields, the Langevin function is

$$L(a) = \frac{1}{3}a - \frac{1}{45}a^3 + \dots,$$

where $a = \frac{\mu}{KT} X$; $X(t)$ is the orientational field at a given moment. The average dipolar moment is

$$\overline{m} = \overline{\mu \cos \theta} = \mu \left[\frac{1}{3} \frac{\mu}{KT} X(t) - \frac{1}{45} \left(\frac{\mu X(t)}{KT} \right)^3 \right].$$

It results the polarization

$$(6) \quad P = n \left[\alpha_0 F + \left(\frac{\mu^2}{3KT} - \frac{\eta^4 X^2}{45(KT)^3} \right) X \right],$$

where F is the internal field.

We cannot neglect the terms higher than one: $X^2, F^2, XF, X^3, \dots$ because in this case the fields are strong.

Knowing that the function of distribution of the dipolar electric moments is

$$f = A \left(1 + \frac{\mu}{KT} X(t) \cos \theta \right); \quad A = \text{constant}$$

and that it must satisfy the Debye equation for the distribution function

$$\xi \frac{\partial f}{\partial t} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \left(KT \frac{\partial f}{\partial \theta} - Mf \right) \right] \quad \text{with } M = -\mu F \sin \theta,$$

we get

$$(7) \quad \frac{\xi}{2KT} \frac{\partial X(t)}{\partial t} + X(t) = F(t) + \frac{\mu}{2KT} \frac{3 \cos^2 \theta - 1}{\cos \theta} X(t) F(t).$$

Because we deal with the variation of the polarization obtained by the orientation of dipolar moments, we can neglect in (6) the polarization by distortion, i.e. $\alpha_0 = 0$ and obtain

$$P = \frac{n\mu^2}{3KT} X(t) - \frac{n\mu^4}{45(KT)^3} X^3(t).$$

As we consider a strong enough field we can appreciate that angle θ between the dipolar moment and the internal field is zero, or it has very small values: $\theta \simeq 0$; $\cos \theta \simeq 1$ and (7) becomes

$$(8) \quad \tau \frac{dX(t)}{dt} + X(t) = F(t) + \frac{\mu}{KT} X(t) F(t).$$

We obtain the following set of equations which describe the dipolar dielectric in strong fields:

$$\tau \frac{dX}{dt} + X = F + \frac{\mu}{KT} XF, \quad D = \epsilon_0 E + P, \quad F = E + \frac{P}{3\epsilon_0}, \quad P = \frac{n\mu^2}{3KT} X - \frac{n\mu^4}{45(KT)^3} X^3.$$

In the following we propose to bring corections of second order to the Debye's classical equation. In this purpose we shall use the following set of equations

$$(9) \quad \begin{cases} \tau \frac{dX}{dt} + X = F + \frac{\mu}{KT} XF, \\ D = \epsilon_0 E + P, & F = E + \frac{P}{3\epsilon_0}, \\ P = \frac{n\mu^2}{3KT} X = aX & \text{where } a = \frac{n\mu^2}{3KT}. \end{cases}$$

Because $F = \frac{D}{3\epsilon_0} + \frac{2E}{3}$ and $X = \frac{D}{a} - \frac{\epsilon_0 E}{a}$ we obtain from the first Eq. (9),

$$(10) \quad \tau \frac{dD}{dt} + \left(1 - \frac{a}{3\epsilon_0}\right) D = \tau \epsilon_0 \frac{dE}{dt} + \epsilon_0 \left(1 + \frac{2a}{3\epsilon_0}\right) E + \frac{\mu}{3KT} \left(\frac{D^2}{\epsilon_0} + DE - 2\epsilon_0 E^2\right).$$

This equation written in SI system, shows the behaviour of a dielectric formed by polar molecules under the influence of a strong external electric field. If we replace here $D = \epsilon_0 E + P_s$ and use the notation (2) we get

$$(11) \quad \frac{dP_s}{dt} + \alpha(T) \left(1 - \frac{\mu E}{KT}\right) P_s - \frac{\mu}{3\epsilon_0 KT} \alpha(T) P_s^2 = \epsilon_0 (\epsilon_s - \epsilon_{0s}) \alpha(T) E.$$

This equation describes the way in which the part of dielectric polarization obtained by the orientation of the dipolar moments in strong fields, modifies in time. The Eq. (11) reduces, in the case of weak fields, to Eq. (5) and it shows the way in which the forming parameters influence the remanent polarization of the electret. If in the solution we replace $T = T_f$; $t = t_f$; $E = E_f$ we obtain the polarization at the beginning of a TSD cycle.

If we put $E = 0$ (the condition of short-circuit of the electret during a TSD cycle), we get from (11) the equation which describes the time variation of an electret's polarization during the termic activated process

$$(12) \quad \frac{dP_s}{dt} + \alpha(T) P_s - \frac{\mu}{3\epsilon_0 KT} \alpha(T) P_s^2 = 0.$$

We suggest that the Eqs. (11) and (12) should replace those used up-to-now (5) and (5') on condition that the intensity of the external electric field E should be greater than 10^4 V/cm .

We continue now by studying the existence and the unicity of the solutions of Eqs. (11) and (12). First let establish the initial conditions. For this we describe a TSD cycle as obtained experimentally. The TSD current, which is the result of the different relaxation mechanisms in short-circuited electrets, is looking as in Fig. 1, where a typical TSD curve for

PMMA is noticed [7]. From the scheme (Fig. 2) time-temperature-electric field we can observe the main characteristic parameters during the forming and storage of the charge of the electret and also during the measurement of the TSD current.

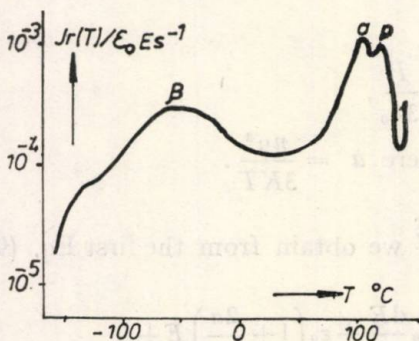


Fig. 1

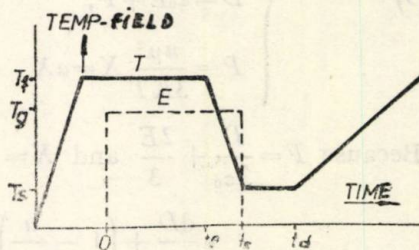


Fig. 2

Let analyse what happens when the electret discharges in a TSD process. Usually the electret is kept a short while $t_d - t_s$ before the measurements TSD begin. We can suppose that after the short-circuit of the electret which is metalized on both sides, the field is zero because the tension on the ammeter which measures the discharge current is neglected. Due to this ideal short-circuit Eq.(11) reduces to (12).

This is a non-linear, of the first order differential equation. Eq. (11) is of Riccati's type and Eq. (12) of Bernoulli's type. For their solving we must indicate an initial condition. This condition for (11) is

$$(13) \quad t_0 = 0; \quad P_0 = P_s(0) = 0.$$

During the cooling of the polymer up to the T_s temperature the electric field maintains its value E . The cooling is made with constant negative rate $s' = dT/dt < 0$.

The solution of Eq. (11) with the initial condition (13) gives the polarization which results after the formation of the thermoelectret as a function of characteristic forming parameters. This is the polarization maintained in the electret after the cooling time due to the dipole orientation. This value of $P_{s0}(T_f, t_f, E_f, t_s, t_d)$ is the initial value for Eq. (12), that describes the variation in time of the polarization in thermostimulated process:

$$(14) \quad t_0 = t_d, \quad P_0 = P_s(t_0) = P_{s0}(T_f, E_f, t_f, t_d, t_s).$$

The solution of Eq. (12) will depend on these parameters by the initial condition (14). The TSD current depends on the „history“ of formation of the electret by this initial condition.

3.1. The existence of the solution for Eq. (11) with the initial condition (14). The equation is of the type $P_s(t) = f(P_s, t)$, with $f(P_s, t)$ a continuous function on the domain

$$(15) \quad D \left\{ \begin{array}{l} |t - t_0| \leq t_d, \\ |P - P_0| \leq P_{sat}. \end{array} \right.$$

According to the existence theorem [12] we obtain a solution $P_s = P_s(t)$ of (11) with the initial condition (13) if for any pair of values (t, P_{s1}) and $(t, P_{s2}) \in D$ the Lipschitz condition is fulfilled,

$$|f(t, P_{s1}) - f(t, P_{s2})| \leq L \text{ where } L > 0 \text{ is a constant.}$$

The solution is defined in the interval $-\delta \leq t \leq +\delta$ where

$$\delta = \min \left\{ t_d, \frac{P_{sat}}{M} \right\} \text{ and } M = \sup |f(P_s, t)|,$$

with $P_s(t) \in D$. In addition $P_s(t)$ is the limit of the row $\{P_{sn}(t)\}$ uniform on $|t - t_0| \leq \delta$.

For Eq. (11) the values L , δ , M are

$$(16) \quad L = \left| \left(\frac{\mu E}{KT} - 1 \right) \alpha(T) + \frac{\mu}{3\epsilon_0 KT} \alpha(T) (P_{s1} + P_{s2}) \right| > 0 \text{ and constant,}$$

$$(17) \quad M = \alpha(T) \epsilon_0 (E_s - \epsilon_\infty)_T E, \quad (18) \quad \delta = \min \{t_d; \tau_{max}\}.$$

From the formula (18) we can see that the chosen experimental conditions must be so, that t_d is smaller than the product between the electric susceptibility and the greatest relaxation time.

3.2. *The existence of the solution for Eq. (12) with the initial condition (14).*

The equation is also of the form $P_s(t) = f_1(P_s, t)$ where $f_1(P_s, t)$ is continuous on the domain

$$(18) \quad D_1 \left\{ \begin{array}{l} |t - t_0| \leq t_{final} - \frac{T_f - T_s + t_d s}{s}, \\ |P_s - P_{s0}| \leq P_{sat} \end{array} \right.$$

and s is the constant rate with which the electret is heating during the thermostimulated discharge. The Lipschitz's condition is fulfilled with the value

$$(19) \quad L_1 = \left| \frac{\mu}{3\epsilon_0 KT} (P_{s1} + P_{s2}) - 1 \right| > 0 \text{ and constant.}$$

The solution is defined in the interval $-\delta_1 \leq t \leq +\delta_1$ where

$$(20) \quad \delta_1 = \min \left\{ t_{final}; \frac{P_{sat}}{M} \right\} \text{ because } M_1 = \sup |f_1(P_s, t)| = \frac{P_{sat}}{\tau_{max}}.$$

From condition (20) we see that the solution exists only if t_{final} is smaller or equal with the greatest relaxation time.

The unicity of both solutions is then demonstrated by reduction ad absurdum.

4. **Conclusions.** 1. As the majority of thermoelectrets are obtained in electric fields, which are stronger than 10^4 V/cm , we propose an equation of Debye's type, which can describe the variation in time of the polarization obtained by dipolar orientation, with a single relaxation time, in the case of strong electric fields.

2. The obtained equation is of Riccati's type and we prove that in the intervals of time and polarization values used experimentally, it admits

a solution and this is unique. In addition, the deduced equation contains the equation employed up-to-now, in weak field approximation.

3. In the case of thermostimulated current, that is of a short-circuited electret, the obtained equation reduces to a Bernoulli's type, having also a unique solution in the interval of time and polarization experimentally used.

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ECUAȚIE TIP DEBYE PENTRU PROCESELE TERMOSTIMULATE ÎN CÂMPURI ELECTRICE INTENSE

(Rezumat)

Cele mai recente lucrări care se ocupă cu procesele thermostimulate în solide, folosesc pentru descrierea acestor procese ecuația clasică Debye, care a fost dedusă pentru studierea comportării gazelor polare sau a soluțiilor cu componenți polari, diluate, în câmpuri electrice slabe.

În această lucrare se propune o corecție a ecuației clasice Debye, corecție care permite studierea proceselor thermostimulate în câmpuri electrice intense. Se particularizează apoi ecuația obținută în cazul scurtcircuitării probei, ceea ce corespunde situației în care se obțin experimental curbele de descărcare thermostimulată (TSD) pentru electreți.





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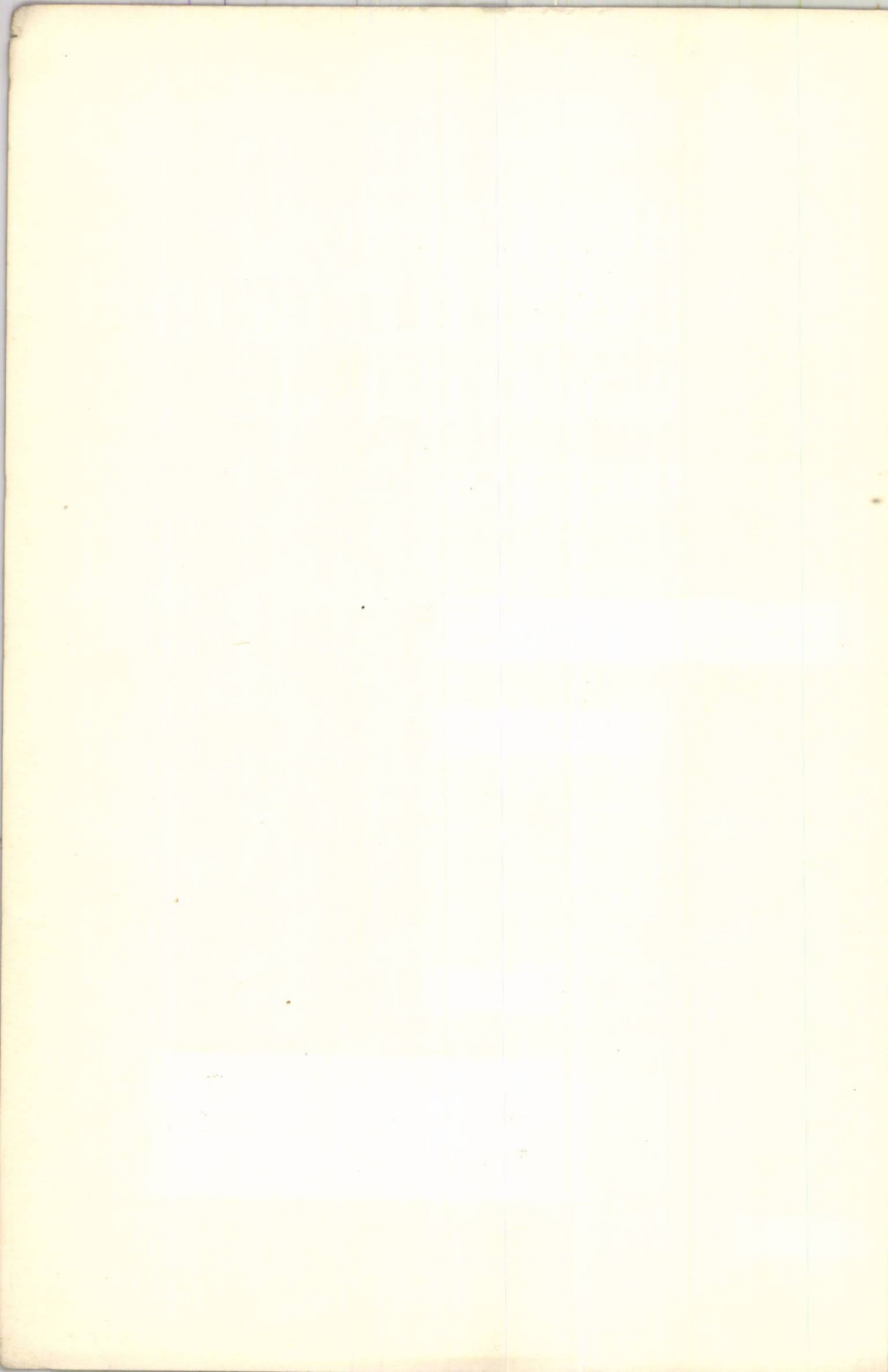
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ВКЛАД В ЗАДАЧИ АСИМПТОТИКИ ЧИСЛА СОБСТВЕННЫХ ЧИСЕЛ**II. О СОБСТВЕННЫХ ЧИСЛАХ И СОБСТВЕННЫХ ФУНКЦИЯХ СПЕКТРАЛЬНОЙ ЗАДАЧИ СВЯЗАННОЙ С ОПЕРАТОРОМ ЛАПЛАСА И ОБЛАСТИ МНОГОГРАННИКОВ ПОРОЖДЕННЫХ СИСТЕМОЙ КОРНЕЙ ТИПА C_n**

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АЛЕКСАНДРУ КЛЙМЕСКУ
к сорокалетней непрерывной научной
деятельности*

1. Факт того, что различные референты, как напр. В. Петровский [10] в своем раферате о работе авторов [11], находят, что выработанный авторами рефлексный метод будет эффективен в решении многих важных инженерных задач, и, как следствие этого, проявили интерес и к последующим работам авторов [12]—[15], поощрил авторов в их стремлении углубить свои исследования выкристаллизовавшиеся в сериале их работ начатых с недавно опубликованной в этом же журнале работе первых трех авторов в сотрудничестве с проф. С. Ниангом [16].

В нижеследующем изложении исследуется проблема о собственных числах и собственных функциях спектральной задачи связанной с оператором Лапласа и с областью многогранников порожденных системой корней типа C_n .

2. Еще Ван дер Варденом [3] на основании простого геометрического метода была проведена классификация полупростых групп Ли. Исходя из трех типов G_2 , B_2 , A_2 в двухмерном евклидовом пространстве E^2 , с помощью расширения им было показано, что существуют четыре бесконечных серии полупростых групп Ли: специальные линейные группы $n+1$ измерения — серия A_n , группы вращений $2n+1$ измерения — серия B_n , комплекс-группы $2n$ измерений — серия C_n , группы вращений $2n$ измерений $n \geq 3$ — серия D_n . Кроме того, имеется еще пять исключительных групп G_2, F_4, E_6, E_7, E_8 .

С другой стороны эти группы допускают геометрическую реализацию и порождают в соответствующих конечномерных пространствах конечные группы отражений — группы Вейля W . Соответствующие им афинные группы Вейля W_a разбивают пространство на симплексы, замыкание каждого из которых является фундаментальной областью для W_a . Каждой серии отвечает свой симплекс, хотя в некоторых случаях он может быть один и тот же для разных серий.

Спектральная задача, связанная с оператором Лапласа в областях, отвечающих полупростым группам Ли A_2, G_2, C_2 , были рассмотрены

в работах [5], [7], случай $n=3$ был рассмотрен в работе [9]. В настоящей работе мы рассмотрим задачу, связанную с оператором Лапласа в области симплекса D , отвечающего серии C_n . Этот симплекс можно описать через фундаментальные веса ω_i [1], вершинами которого будут 0 и ω_i/n_i , где n_i — координаты максимального корня [1]. В случае серии C_n

$$(1) \quad \frac{\omega_i}{n_i} = \frac{a}{2} (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i),$$

где $a > 0$, а ε_i канонический базис в E^n . Кроме того, симплекс D можно выделить системой неравенств

$$(2) \quad x_i - x_{i+1} > 0, \quad x_n > 0, \quad 2x_1 < a.$$

Из формулы (1) находим, что

$$(3) \quad \text{mes}_n(D) = \frac{a^n}{2^n n!}.$$

меры граней, лежащих на гиперплоскостях $H_{i+2} = \{\bar{x} \in E^n; x_i - x_{i+1} = 0\}$, равны

$$(4) \quad \frac{a^{n-1} \sqrt{2}}{2^{n-1} (n-1)!},$$

а меры площадей граней, лежащих на гиперплоскостях $H_1 = \{\bar{x} \in E^n; x_n = 0\}$ и $H_2 = \{\bar{x} \in E^n; 2x_1 = a\}$, равны

$$(5) \quad \frac{a^{n-1}}{2^n (n-1)!}.$$

Обозначим еще зеркальное отражение относительно H_i через S_{H_i} . Тогда $S_{H_{i+2}}(\bar{x}_i)$ является просто транспозицией: это есть перестановка x_i и x_{i+1} , а $S_{H_1}(\bar{x})$ меняет лишь знак перед x_n . Что касается $S_{H_2}(\bar{x})$, то это есть композиция отображений, которое меняет знак перед x_1 и переноса на вектор $(a, 0, \dots, 0)$. Таким образом, группа Вейля W является полупрямым произведением группы перестановок чисел $(1, 2, \dots, n)$, которую мы обозначим через J^n , и группы $(Z/2Z)^n$ отображений $x_i \rightarrow (\pm 1)x_i$. Ее порядок равен $2^n n!$. Афинная группа Вейля $W_a = W \times Q$, где Q — решетка

$$(6) \quad Q = aI\bar{s}, \quad \bar{s} \in Z^n,$$

причем Z — группа целых чисел, I — единичная матрица размера $n \times n$.

3. Рассмотрим в области D следующую граничную задачу

$$(7) \quad (-1)^{p-1} \Delta_n^p u(\bar{x}, \bar{\xi}) + \lambda u(\bar{x}, \bar{\xi}) = \delta_{\bar{\xi}}, \quad \bar{x} \in D,$$

$$k_i \partial_{\vec{v}_i}^{2r+1} u + r_i \partial_{\vec{v}_i}^{2r} u = 0, \quad \bar{x} \in \bar{D} \cap H_i,$$

где $\vec{v}_i$ внутренняя нормаль к $\bar{D} \cap H_i$, k_i и r_i неотрицательные целые числа под условием $k_i + r_i = 1$, $r = 0, 1, \dots, p-1$; $p = \left[\frac{n}{2} \right] + 1$, $[x]$ — целая часть числа x , $\delta_{\bar{\xi}}$ — мера Дирака, сосредоточенная в точке $\bar{\xi} \in D$ и, на-

конец, Δ_n оператор Лапласа (по $\vec{x}$) в E^n . Предположим еще что $k_3=k_4=\dots=k_{n-1}$. Это условие обеспечивает [8] гладкость решения вплоть до границы, так что отпадает необходимость формулировать условия на ребрах многогранника.

Обозначим еще через $\vec{\xi}_j^{(\sigma)}$ ($\sigma \in J^n$) вектор с компонентами

$$(8) \quad \{(-1)^{j_1} \xi_{\sigma(1)}, (-1)^{j_2} \xi_{\sigma(2)}, \dots, (-1)^{j_n} \xi_{\sigma(n)}\},$$

через $\varepsilon(\sigma)$ — сигнатуру подстановки $\sigma \in J^n$. Пусть далее $|\vec{s}| = s_1 + s_2 + \dots + s_n$ — норма мультииндекса $\vec{s}$, $|\vec{j}| = j_1 + j_2 + \dots + j_n$, причем j_k принимает независимо друг от друга значения 0 или 1. Положим еще $p_{10}=1$, а $p_{01}=-1$. При этих предположениях и обозначениях справедлива такая

Теорема 1. Пусть $T(\|\vec{x}-\vec{\xi}\|, \lambda)$ есть регулярное на бесконечности, инвариантное относительно группы вращений фундаментальное решение оператора $(-1)^{p-1} \Delta_n^p + \lambda I$. Тогда, если $\mathcal{M} \lambda \neq 0$,

$$(9) \quad u(\vec{x}, \vec{\xi}, \lambda) = T(\|\vec{x}-\vec{\xi}\|, \lambda) + \sum_{s \in Z^n} \sum_{j_1, j_2, \dots, j_n=0}^1 \sum_{\sigma \in J^n} (p_{k_1 r_1})^{|\vec{s}|+|\vec{j}|} (p_{k_2 r_2})^{|\vec{s}|} (p_{k_3 r_3})^{\frac{1-\varepsilon(\sigma)}{2}} T(\|\vec{x}-a I \vec{s}-\vec{\xi}_j^{(\sigma)}\|, \lambda)$$

есть решение граничной задачи (7). Ряд в правой части формулы (9) сходится абсолютно и равномерно в $\bar{D}$ вместе со своими производными любого порядка. $(\Sigma)'$ означает, что $\vec{s}=\vec{0}$ и $j_k=0$ ($k=1, 2, \dots, n$) в суммировании не участвуют.

Доказательство. При $\mathcal{M} \lambda \neq 0$ регулярное фундаментальное решение экспоненциально убывает на бесконечности и ряд в формуле (9) можно почленно дифференцировать в $\bar{D}$. Поэтому необходимо лишь показать, что относительно любой части границы $\bar{D} \cap H_i$ функцию $u(\vec{x}, \vec{\xi}, \lambda)$ можно представить в виде суммы слагаемых

$$(10) \quad T(\|\vec{x}-\vec{\xi}_\mu\|, \lambda) + p_{k_i r_i} T(\|\vec{x}-S_{H_i}(\vec{\xi}_\mu)\|, \lambda).$$

С этой целью необходимо рассмотреть целочисленные унимодулярные подстановки, ассоциированные с S_{H_i} .

В случае границ $\bar{D} \cap H_{i+2}$ ассоциированная с $S_{H_{i+2}}$ унимодулярная подстановка имеет вид

$$(11) \quad \alpha_{i+2} = \begin{cases} s_i \leftrightarrow s_{i+1}, & j_i \leftrightarrow j_{i+1}; \\ s_k \leftrightarrow s_k, & k \neq i, i+1, \quad j_{\tilde{p}} \rightarrow j_{\tilde{p}}, \quad \tilde{p} \neq i, i+1; \\ & \sigma(i) \leftrightarrow \sigma(i+1). \end{cases}$$

Так как по $\vec{j}$ подстановка α_{i+2} является транспозицией, то сигнатура такой подстановки равна -1 , и представление в виде сумм (10) имеет место.

В случае границы $\bar{D} \cap H_1$ имеем

$$(12) \quad \alpha_1 = \begin{cases} s_i \rightarrow s_i, & i \neq n, \quad j_k \rightarrow j_k \quad (k \neq n); \\ s_n \rightarrow -s_n, & j_n \rightarrow j_{n+1} \pmod{2}; \\ & \sigma(i) \rightarrow \sigma(i). \end{cases}$$

И, наконец, в случае границы $\bar{D} \cap H_2$ соответствующая унимодулярная подстановка будет

$$(13) \quad \alpha_2 = \begin{cases} s_i \rightarrow s_i \ (i \neq 1), & j_k \rightarrow j_k \ (k \neq 1); \\ s_1 \rightarrow s_1 + 1, & j_1 \rightarrow j_1 + 1 \pmod{2}; \\ \sigma(i) \rightarrow \sigma(i) \end{cases}$$

и мы видим, что при этой подстановке первый член в сумме (9) не меняется. Теорема доказана.

С л е д с т в и е. Полная система собственных функций граничной задачи

$$(14) \quad \begin{aligned} (-1)^{p-1} \Delta_n \varphi(\vec{\xi}) + \lambda \varphi(\vec{\xi}) &= 0, \quad \vec{\xi} \in D, \\ k_i \partial_{\vec{v}_i}^{2r_i+1} \varphi + r_i \partial_{\vec{v}_i}^{2r_i} \varphi &= 0, \quad \vec{\xi} \in \bar{D} \cap H, \end{aligned}$$

имеет вид

$$(15) \quad \varphi_s^-(\vec{\xi}) = \kappa_s^- \sum_{\sigma \in J^n} \sum_{j_1, j_2, \dots, j_n} (p_{k_1 r_1})^{|\vec{j}|} (p_{k_2 r_2})^{\frac{1-\varepsilon(\sigma)}{2}} \exp \pi i (A_S^{-1}, \vec{\xi}_{\vec{j}}^{(\sigma)}),$$

$$(A = aI),$$

причем собственной функции $\varphi_s^-(\vec{\xi})$ отвечает собственное значение

$$(16) \quad \lambda_s^- = \|\pi A^{-1} \vec{s}\|^{2p},$$

а

$$(17) \quad \kappa_s^- = \prod_{j=1}^n \{1 + p_{k_1 r_1} p_{k_2 r_2} (-1)^{j_j}\}.$$

Д о к а з а т е л ь с т в о. Спектральная задача (14) сводится к изучению на действительной оси λ полюсов функции $u(\vec{x}, \vec{\xi}, \lambda)$ и вычетов в соответствующих полюсах. Запишем фундаментальное решение в виде интеграла Фурье. Имеем

$$(18) \quad T(\|\vec{x} - \vec{\xi}\|, \lambda) = \int_{E^n} \frac{\exp 2\pi i (\vec{x} - \vec{\xi}, \vec{\omega})}{\lambda - \|2\pi \vec{\omega}\|^{2p}} d\vec{\omega},$$

Подставляя это в формулу (9) и суммируя в отдельности по четным и нечетным индексам s_j , получаем

$$(19) \quad u(\vec{x}, \vec{\xi}, \lambda) = \sum (p_{k_1 r_1})^{|\vec{j}|} (p_{k_2 r_2})^{\frac{1-\varepsilon(\sigma)}{2}} \int_{E^n} \frac{\kappa(\vec{\omega}) \exp 2\pi i (\vec{x} - 2a\vec{s} - \vec{\xi}_{\vec{j}}^{(\sigma)}, \vec{\omega})}{\lambda - \|2\pi \vec{\omega}\|^{2p}} d\vec{\omega},$$

где

$$(20) \quad \kappa(\vec{\omega}) = \prod_{j=1}^n (1 + p_{k_1 r_1} p_{k_2 r_2} \exp 2i\pi \omega_j).$$

Поскольку $2p > n$, то в формуле (19) можно воспользоваться формулой суммирования Пуассона. Таким путем мы сразу же приходим к спектральному разложению

$$(21) \quad u(\vec{x}, \vec{\xi}, \lambda) = \frac{1}{(2a)^n} \sum_{\vec{s} \in \mathbb{Z}^n} \frac{\exp \frac{2\pi i}{2a} (\vec{x}, \vec{s})}{\lambda - \left\| \frac{\pi \vec{s}}{a} \right\|^{2p}} \varphi_{\vec{s}}(\vec{\xi}).$$

Отсюда немедленно следует справедливость нашего утверждения.

4. Рассмотрим более подробно собственные функции $\varphi_{\vec{s}}(\vec{\xi})$. Прежде всего, в формуле (15) можно выполнить суммирование по j_k . Таким путем получим

$$(22) \quad \varphi_{\vec{s}}(\vec{\xi}) = \begin{cases} 2^{n-m} \chi_{\vec{s}} \sum_{\sigma \in J^n} (p_{k_{\sigma r_{\sigma}}})^{\frac{1-\varepsilon(\sigma)}{2}} \prod_{j=1}^n \cos \frac{2\pi s_j \xi_{\sigma(j)}}{2a}, & \text{при } p_{k_{r_1}} = 1, \\ (2i)^n \chi_{\vec{s}} \sum_{\sigma \in J^n} (p_{k_{\sigma r_{\sigma}}})^{\frac{1-\varepsilon(\sigma)}{2}} \prod_{j=1}^n \sin \frac{2\pi s_j \xi_{\sigma(j)}}{2a}, & \text{при } p_{k_{r_1}} = -1, \end{cases}$$

где m обозначает число ненулевых индексов s_j . Поскольку ни $\chi_{\vec{s}}$, ни $\|\vec{s}\|$ не меняются от перемены знаков перед компонентами s_j , то в билинейном разложении вместо $\exp \frac{\pi i}{2a} (\vec{x}, \vec{s})$ можно писать

$$\begin{aligned} & 2^{n-m} \prod_{j=1}^n \cos \frac{2\pi s_j x_j}{2a}, \quad \text{при } p_{k_{r_1}} = 1, \\ & (2i)^n \prod_{j=1}^n \sin \frac{2\pi s_j x_j}{2a}, \quad \text{при } p_{k_{r_1}} = -1. \end{aligned}$$

И теперь в билинейном разложении можно вести суммирование по направленному множеству индексов $s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0$. Таким путем мы приходим к симметричному билинейному разложению

$$(23) \quad u(\vec{x}, \vec{\xi}, \lambda) = \sum_{s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0} \frac{\psi_{\vec{s}}(\vec{x}) \psi_{\vec{s}}(\vec{\xi})}{\lambda - \left\| \frac{\pi \vec{s}}{a} \right\|^{2p}},$$

где

$$(24) \quad \psi_{\vec{s}}(\vec{\xi}) = \sqrt{\frac{2^{n-2m} n!}{a^n}} \chi_{\vec{s}} \sum_{\sigma \in J^n} (p_{k_{\sigma r_{\sigma}}})^{\frac{1-\varepsilon(\sigma)}{2}} \prod_{j=1}^n \cos \frac{2\pi s_j \xi_{\sigma(j)}}{2a}, \quad \text{при } p_{k_{r_1}} = 1$$

и аналогичное выражение для случая $p_{k_{r_1}} = -1$.

Из формулы (17), кроме того, вытекает, что либо все s_j — четные, если граничные условия на $\bar{D} \cap H_1$ и $D \cap H_2$ одинаковы, либо все s_j нечетные, если условия на $\bar{D} \cap H_1$ и $\bar{D} \cap H_2$ не одинаковы.

Таким образом, получаем:

1) в случае задачи Неймана

$$(25) \quad \begin{aligned} \psi_{\vec{s}}(\vec{\xi}) &= \frac{2^{n-m} \sqrt{n!}}{\sqrt{a^n}} \sum_{\sigma \in J^n} \prod_{j=1}^n \cos \frac{2\pi s_j \xi_{\sigma(j)}}{a}, \\ \lambda_{\vec{s}} &= \left\| \frac{2\pi}{a} \vec{s} \right\|^{2p}, \quad s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0; \end{aligned}$$

2) в случае задачи Дирихле

$$(26) \quad \begin{aligned} \psi_{\vec{s}}(\vec{\xi}) &= \frac{2^n \sqrt{n!}}{\sqrt{a^n}} \det \left\{ \sin \frac{2\pi s_j \xi_i}{a} \right\}, \\ \lambda_{\vec{s}} &= \left\| \frac{2\pi \vec{s}}{a} \right\|^{2p}, \quad s_n > s_{n-1} > \dots > s_1 > 0. \end{aligned}$$

В случае смешанных задач имеется еще 6 возможностей. Соответствующие серии собственных функций мы будем называть сериями $(p_{k_1 r_1}, p_{k_2 r_2}, p_{k_3 r_3})$. Итак, получаем:

1) для серии $(-1, -1, 1)$

$$(27) \quad \begin{aligned} \psi_{\vec{s}}(\vec{\xi}) &= \frac{2^n \sqrt{n!}}{\sqrt{a^n}} \sum_{\sigma \in J^n} \prod_{j=1}^n \sin \frac{\pi(2s_j + 1) \xi_{\sigma(j)}}{a}, \\ \lambda_{\vec{s}} &= \left\| \frac{\pi}{a} (2\vec{s} + \vec{1}) \right\|^{2p}, \quad s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0; \end{aligned}$$

2) для серии $(1, -1, 1)$

$$(28) \quad \begin{aligned} \psi_{\vec{s}}(\vec{\xi}) &= \frac{2^{n-m} \sqrt{n!}}{\sqrt{a^n}} \sum_{\sigma \in J^n} \prod_{j=1}^n \cos \frac{\pi(2s_j + 1) \xi_{\sigma(j)}}{a}, \\ \lambda_{\vec{s}} &= \left\| \frac{\pi}{a} (2\vec{s} + \vec{1}) \right\|^{2p}, \quad s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0; \end{aligned}$$

3) для серии $(-1, -1, 1)$

$$(29) \quad \begin{aligned} \psi_{\vec{s}}(\vec{\xi}) &= \frac{2^n \sqrt{n!}}{\sqrt{a^n}} \sum_{\sigma \in J^n} \prod_{j=1}^n \sin \frac{2\pi s_j \xi_{\sigma(j)}}{a}, \\ \lambda_{\vec{s}} &= \left\| \frac{2\pi}{a} \vec{s} \right\|^{2p}, \quad s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0; \end{aligned}$$

4) для серии $(-1, 1, -1)$

$$(30) \quad \begin{aligned} \psi_{\vec{s}}(\vec{\xi}) &= \frac{2^n \sqrt{n!}}{\sqrt{a^n}} \det \left\{ \sin \frac{\pi(2s_j + 1) \xi_i}{a} \right\}, \\ \lambda_{\vec{s}} &= \left\| \frac{\pi}{a} (2\vec{s} + \vec{1}) \right\|^{2p}, \quad s_n > s_{n-1} > \dots > s_1 \geq 0; \end{aligned}$$

5) для серии $(1, -1, -1)$

$$(31) \quad \psi_{\vec{s}}(\vec{\xi}) = \frac{2^{n-m} \sqrt{n!}}{\sqrt{a^n}} \det \left\{ \cos \frac{\pi(2s_j+1)}{a} \xi_i \right\},$$

$$\lambda_{\vec{s}} = \left\| \frac{\pi}{a} (2\vec{s} + \vec{1}) \right\|^{2p}, \quad s_n > s_{n-1} > \dots > s_1 \geq 0;$$

6) для серии (1, 1, -1)

$$(32) \quad \psi_{\vec{s}}(\vec{\xi}) = \frac{2^{n-m} \sqrt{n!}}{\sqrt{a^n}} \det \left\{ \cos \frac{2\pi s_j \xi_i}{a} \right\},$$

$$\lambda_{\vec{s}} = \left\| \frac{2\pi}{a} \vec{s} \right\|^{2p}, \quad s_n > s_{n-1} > \dots > s_1 \geq 0.$$

5. Теперь изучим асимптотику при больших λ спектральной функции $N(\lambda)$. Наша задача свелась, как мы видим, к задаче подсчета целых точек внутри сферического сектора. В случае, если условия на $\bar{D} \cap H_1$ и $\bar{D} \cap H_2$ разные, имеем

$$(33) \quad N(\lambda) = \left\{ \text{card } \vec{s}; \sum_{i=1}^n \left(s_i + \frac{1}{2} \right)^2 \leq \frac{a^2 \lambda^{\frac{1}{p}}}{(2\pi)^2}, s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0, s_i \in Z, \lambda \in E \right\},$$

причем строгие равенства достигаются в случае, если на $\bar{D} \cap H_{i+2}$ выполняются условия Неймана, а нестрогие — в случае задачи Дирихле. Из формул (33) сразу же находим главный член асимптотики спектральной функции

$$(34) \quad N(\lambda) = \left(\frac{a \lambda^{\frac{1}{2p}}}{2\pi} \right)^n \frac{v_n}{2^n n!} + o(\lambda^{\frac{n}{2p}}),$$

где v_n — величина объема единичного шара в E^n .

Для нахождения второго точного члена асимптотики необходимо еще найти половину точек на сечениях $s_i = s_{i+1}$. Так как эти сечения образуют с координатной сеткой угол α в 45 градусов, то второй член асимптотики будет

$$(35) \quad \pm \frac{1}{2} \left(\frac{a \lambda^{\frac{1}{2p}}}{2\pi} \right)^{n-1} \frac{v_{n-1}(n-1)}{2^{n-1} \sqrt{2}(n-1)!}.$$

Итак, с учетом формул (33), (34), (35) имеем

$$(36) \quad N(\lambda) = \frac{v_n \text{mes}_n D}{(2\pi)^n} \lambda^{\frac{n}{2p}} + \frac{v_{n-1} \sum \rho_{i,i} \text{mes}_{n-1}(\bar{D} \cap H_i)}{4(2\pi)^{n-1}} \lambda^{\frac{n-1}{2p}} + \rho_n(\lambda).$$

Если же на $\bar{D} \cap H_1$ и $\bar{D} \cap H_2$ выполняются различные граничные условия, то

$$(37) \quad N(\lambda) = \left\{ \text{card } \vec{s}; \|\vec{s}\|^2 \leq \frac{a^2 \lambda^{\frac{1}{p}}}{(2\pi)^2}, s_n \geq s_{n-1} \geq \dots \geq s_1 \geq 0; s_i \in Z, \lambda \in E \right\},$$

так что здесь необходимо учесть еще число целых точек на сечениях $s_1=0$. Оно равно

$$(38) \quad \frac{v_{n-1} \lambda^{\frac{n-1}{2p}}}{(2\pi)^n 2^{n-1} (n-1)!},$$

что с учетом формулы (5) снова приводит к формуле (36).

Дальнейшее уточнение асимптотики спектральной функции связано с известными проблемами нахождения второго члена асимптотики числа целых точек в шаре. Как известно [2], при $n \geq 5$ мы имеем точные по порядку оценки второго члена асимптотики, а именно o -проблема утверждает, что в формуле (36) остаток имеет вид $o\left(\lambda^{\frac{n}{2p}-1}\right)$. Если же $n \leq 4$, то здесь нет точных по порядку оценок. Они неоднократно уточнялись и остаток в формуле (36) будет иметь такие порядки

$$\rho_n(\lambda) = \begin{cases} O\left(\lambda^{\frac{13}{40p}}\right) & \text{при } n=2 \quad (\text{Хуа Ло-ген [6]}); \\ O\left(\lambda^{\frac{2}{3p}} \ln^6 \lambda\right) & \text{при } n=3 \quad (\text{Виноградов И. М. [4]}); \\ O\left(\lambda^{\frac{1}{p}} \ln^{\frac{2}{3}} \lambda\right) & \text{при } n=4 \quad (\text{Вальфиш А. З. [2]}). \end{cases}$$

Во всяком случае остаток в формуле (36) имеет порядок $o\left(\lambda^{\frac{n-1}{2p}}\right)$.

6. *Примечания.* 1°. В последующих работах будут изучены спектральные задачи связанные с операторами Лапласа для многогранников соответствующих — по терминологии Н. Бурбаки [1] — каждой из других чем рассмотренная в [16] и в настоящей работе полупростых групп Ли. Будет изложена асимптотика спектральной функции причем нахождением второго члена этой асимптотики доказывается шаг за шагом гипотеза Г. Вейля.

2°. Впоследствии намечается углубить, в рамках разрабатываемых проблем, некоторые вопросы поставленные в работах проф. Ал. К. Климеску и Ал. Энгеля [17], [18], соответственно.

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CONTRIBUȚII LA PROBLEMA DE EVALUĂRI ASIMPTOTICE A AUTOVALORIILOR

II. Determinarea autovalorilor și autofuncțiilor problemei spectrale referitoare la operatorul Laplace în domeniul poliedrelor generate de sistemul de rădăcini de tipul C_n

(Rezumat)

În continuarea studiului [16] se rezolvă problema spectrală referitoare la operatorul Laplace în domeniul unui simplex D corespunzător seriei infinite C_n de grupuri semisimple Lie cu $2n$ dimensiuni (după clasificarea lui B. van der Waerden [3]). Într-o lucrare viitoare se va da evaluarea asimptotică a funcției spectrale $N(\lambda)$ și se va determina cel de al doilea termen al unei astfel de evaluări asimptotice, demonstrind ipoteza corespunzătoare formulată la timpul său de H. Weil (a se vedea N. Bourbaki [1]).

OTHER ELEMENTARY PROPERTIES OF THE MAPPINGS OF TOPOLOGICAL SPACES

BY

GH. COSTOVICI

This Note is connected with [1]; both contain properties of the mappings or of the compositions of mappings of topological spaces.

Definitions. Let $f: X \rightarrow Y$ be a single valued function, X and Y being topological spaces.

(1) f is strong continuous (denoted as str. c.) $\Leftrightarrow f(\text{Cl}(A)) \subseteq f(A)$ (Cl denotes the closure operator) for all $A \subseteq X \Leftrightarrow f^{-1}(B)$ is open for all $B \subseteq Y \Leftrightarrow f^{-1}(B)$ is closed for all $B \subseteq Y \Leftrightarrow f^{-1}(B)$ is open and closed for all $B \subseteq Y$ (all equivalences are proved in [2]).

Remark. f str.c. $\Rightarrow f$ continuous.

(2) f is weakly continuous (denoted as w.c.) at $x \in X \Leftrightarrow$ for $f(x) \in D_2$ open in Y , there is an open set D_1 in X such that $x \in D_1$ and $f(D_1) \subseteq \text{Cl}(D_2)$.

f is w.c. $\Leftrightarrow f$ is w.c. at each of the points in $X \Leftrightarrow$ for D_2 open in Y , $f^{-1}(D_2) \subseteq \text{Int}(f^{-1}(\text{Cl}(D_2)))$ (Int denotes the interior operator and the last equivalence is proved in [3]).

(3) f is w.c. $\Leftrightarrow$ when D_2 is open in Y , then $f^{-1}(\text{Fr}(D_2))$ is closed in X , where Fr denotes the frontier operator (this definition is in [3]).

Remark. It is proved in [3] that f is continuous $\Leftrightarrow f$ is both w.c. and w*.c.

(4) A set $A \subseteq X$ is semi-open $\Leftrightarrow$ there exists an open set $D \subseteq X$ such that $D \subseteq A \subseteq \text{Cl}(D)$.

f is semi-continuous (written sem.c.) $\Leftrightarrow$ for D_2 open in Y , then $f^{-1}(D_2)$ is semi-open $\Leftrightarrow$ for $f(x) \in D_2$ open in Y , there exists an A semi-open in X such that $x \in A$ and $f(A) \subseteq D_2$ (the last equivalence is proved in [4]).

Remark. f continuous $\Leftrightarrow f$ sem.c.

(5) f is semi-weakly continuous (written sem.w.c.) $\Leftrightarrow$ for $f(x) \in D_2$ open in Y , then there exists A semi-open in X such that $x \in A$ and $f(A) \subseteq \text{Cl}(D_2)$ (this definition perhaps is new).

Remarks. 1) f continuous $\Rightarrow f$ w.c. $\Rightarrow f$ sem.w.c.

2) f continuous $\Rightarrow f$ sem.c. $\Rightarrow f$ sem.w.c.

(6) f is open (closed) $\Leftrightarrow$ for every A open (closed) in X , $f(A)$ is open (closed) in Y .

Proposition. If $X_i, i=\overline{1,3}$ are topological spaces and $f_j : X_j \rightarrow X_{j+1}$ $j=\overline{1,2}$ are single valued functions, then, we have :

- (1) f_1 continuous and f_2 str.c. $\Rightarrow f_2 f_1$ str.c.,
- (2) $f_2 f_1$ str.c., f_1 open or closed and surjection $\Rightarrow f_2$ str.c.,
- (3) $f_2 f_1$ str.c. and f_2 injection $\Rightarrow f_1$ str.c.,
- (4) $f_2 f_1$ w.c. and f_1 open surjection $\Rightarrow f_2$ w.c.,
- (5) $f_2 f_1$ w.c. and f_1 closed surjection $\Rightarrow f_2$ w.c.,
- (6) f_1 w.c. and f_2 continuous $\Rightarrow f_2 f_1$ w.c.,
- (7) f_1 continuous and f_2 w.c. $\Rightarrow f_2 f_1$ w.c.,
- (8) $f_2 f_1$ sem.c. and f_1 open continuous surjection $\Rightarrow f_2$ sem.c.,
- (9) $f_2 f_1$ sem.c. and f_2 open injection $\Rightarrow f_1$ sem.c.,
- (10) f_1 sem.c. and f_2 continuous $\Rightarrow f_2 f_1$ sem.c.,
- (11) f_1 homeomorphism and f_2 sem.c. $\Rightarrow f_2 f_1$ sem.c.,
- (12) f_1 sem.c. and f_2 w.c. $\Rightarrow f_2 f_1$ sem.w.c.,
- (13) f_1 sem.w.c. and f_2 continuous $\Rightarrow f_2 f_1$ sem.w.c.,
- (14) $f_2 f_1$ sem.w.c. and f_1 open continuous surjection $\Rightarrow f_2$ sem.w.c.

Proof. Before make the proof, we recall that if g is injection then $g^{-1}(g(A))=A$ and if g is surjection then $g(g^{-1}(A))=A$.

- (1) $(f_2 f_1)^{-1}(C) = f_1^{-1} f_2^{-1}(C)$ is open for all $C \subseteq X_3$.
- (2) Let f_1 be open surjection. $(f_2 f_1)^{-1}(C) = f_1^{-1} f_2^{-1}(C)$ open $\Rightarrow f_1 f_1^{-1} f_2^{-1}(C) = f_2^{-1}(C)$ open for all $C \subseteq X_3$. Analogously if f_1 is closed surjection.
- (3) $(f_2 f_1)^{-1}(f_2(B)) = f_1^{-1} f_2^{-1} f_2(B) = f_1^{-1}(B)$ is open for all $B \subseteq X_2$.
- (4) If $f_2(x_2) \in D_3$ open in X_3 , then $\exists x_1 \in X_1$ and $f_1(x_1) = x_2$. $f_2 f_1(x_1) = f_2(x_2) \in D_3$ open in $X_3 \Rightarrow \exists D_1$ open in X_1 such that $x_1 \in D_1$ and $f_2 f_1(D_1) \subseteq \text{Cl}(D_3)$. Denoting $f_1(D_1) = D_2$ open in X_2 , we have $x_2 \in D_2$ and $f_2(D_2) \subseteq \text{Cl}(D_3)$.
- (5) If D_3 is open in X_3 , then $(f_2 f_1)^{-1}(\text{Fr}(D_3)) = f_1^{-1} f_2^{-1} \text{Fr}(D_3)$ closed $\Rightarrow f_1 f_1^{-1} f_2^{-1} \text{Fr}(D_3) = f_2^{-1} \text{Fr}(D_3)$ closed.
- (6) Let $f_2 f_1(x_1) \in D_3$ where D_3 is open in X_3 . We denote $f_1(x_1) = x_2$. Then $f_2(x_2) \in D_3$ and then $\exists D_2$ open in X_2 so that $x_2 \in D_2$ and $f_2(D_2) \subseteq D_3$. $f_1(x_1) = x_2 \in D_2 \Rightarrow \exists D_1$ open in X_1 such that $x_1 \in D_1$ and $f_1(D_1) \subseteq \text{Cl}(D_2) \Rightarrow f_2 f_1(D_1) \subseteq f_2 \text{Cl}(D_2) \subseteq \text{Cl} f_2(D_2) \subseteq \text{Cl}(D_3)$.
- (7) If D_3 is open in X_3 , then $(f_2 f_1)^{-1}(\text{Fr}(D_3)) = f_1^{-1} f_2^{-1} \text{Fr}(D_3)$ is closed in X_1 .

(8) Firstly we give another proof of the Theorem 9 from [4]; if f is open and continuous and if the set A is semi-open, then $f(A)$ is semi-open. Namely, if A is semi-open, then there exists D open so that $D \subseteq A \subseteq \text{Cl}(D) \Rightarrow f(D) \subseteq f(A) \subseteq f(\text{Cl}(D)) \subseteq \text{Cl}(f(D))$. We apply this Theorem at the proof of (8). If D_3 is open in X_3 , then $(f_2 f_1)^{-1}(D_3) = f_1^{-1} f_2^{-1}(D_3)$ is semi-open in X_1 . Then $f_1 f_1^{-1} f_2^{-1}(D_3) = f_2^{-1}(D_3)$ is semi-open in X_2 .

(9) If D_2 is open in X_2 , then $f_2(D_2)$ is open in X_3 . Then $(f_2 f_1)^{-1}(f_2(D_2)) = f_1^{-1} f_2^{-1} f_2(D_2) = f_1^{-1}(D_2)$ is semi-open in X_1 .

(10) If D_3 is open in X_3 , then $(f_2 f_1)^{-1}(D_3) = f_1^{-1} f_2^{-1}(D_3)$ is semi-open in X_1 .

(11) If D_3 is open in X_3 , there exists D_2 open in X_2 such that $D_2 \subseteq f_2^{-1}(D_3) \subseteq \text{Cl}(D_3)$. Then $f_1^{-1}(D_2) \subseteq f_1^{-1} f_2^{-1}(D_3) = (f_2 f_1)^{-1}(D_3) \subseteq f_1^{-1} \text{Cl}(D_2) \subseteq \text{Cl} f_1^{-1}(D_2)$ where $f_1^{-1}(D_2)$ is open in X_1 and then $(f_2 f_1)^{-1}(D_3)$ is semi-open in X_1 .

(12) Let $f_2 f_1(x_1) \in D_3$ where D_3 is open in X_3 . We denote $f_1(x_1) = x_1$. Then, $f_2(x_2) \in D_3$ open in X_3 and then there exists D_2 open in X_2 so that $x_2 \in D_2$ and $f_2(D_2) \subseteq \text{Cl}(D_3)$. $f_1(x_1) = x_2 \in D_2$ open in $X_2 \Rightarrow \exists A$ semi-open in X_1 so that $x_1 \in A$ and $f_1(A) \subseteq D_2 \Rightarrow f_2 f_1(A) \subseteq f_2(D_2) \subseteq \text{Cl}(D_3)$.

(13) Let $f_2 f_1(x_1) \in D_3$ where D_3 is open in X_3 . We denote $f_1(x_1) = x_1$. $f_2(x_2) \in D_3$ open in $X_3 \Rightarrow \exists D_2$ open in X_2 such that $x_2 \in D_2$ and $f_2(D_2) \subseteq D_3$. $f_1(x_1) = x_2 \in D_2$ open in $X_2 \Rightarrow \exists A$ semi-open in X_1 so that $x_1 \in A$ and $f_1(A) \subseteq \text{Cl}(D_2) \Rightarrow f_2 f_1(A) \subseteq f_2 \text{Cl}(D_2) \subseteq \text{Cl} f_2(D_2) \subseteq \text{Cl}(D_3)$.

(14) Let $f_2(x_2) \in D_3$ with D_3 open in X_3 . Then there exists $x_1 \in X_1$ so that $f_1(x_1) = x_2$ and, $f_2 f_1(x_1) = f_2(x_2) \in D_3$ open in X_3 . Then $\exists A$ semi-open in X_1 so that $x_1 \in A$ and $f_2 f_1(A) \subseteq \text{Cl}(D_3)$. $f_1(A)$ is semi-open in X_2 (see the proof of (8)) and we have $x_2 \in f_1(A)$ and $f_2(f_1(A)) \subseteq \text{Cl}(D_3)$.

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ALTE PROPRIETĂȚI ELEMENTARE ALE APLICAȚIILOR DE SPAȚII TOPOLOGICE

(Rezumat)

Se demonstrează mai multe proprietăți pentru aplicații de spații topologice sau pentru compuse ale lor — aplicațiile înseși fiind supuse la diferite condiții — proprietăți care decurg respectiv din proprietăți ale compuselor sau ale membrilor componenți.

SUR L'ESPACE DES VECTEURS LIBRES ASSOCIÉS À UNE VARIÉTÉ LINÉAIRE D'UN ESPACE AFFINE

PAR

DAN BORSȘ

1. Le but de cette Note est de continuer les idées exposées dans [1] en définissant l'espace des vecteurs libres associés à une variété linéaire donnée dans un espace affine.

Partout dans ce qu'il suit, les définitions et les notations sont celles de [1].

2. Soit $(\mathcal{A}, C^n)$ un espace affine complexe, M_0 un point quelconque fixé dans $\mathcal{A}$ et S un espace vectoriel de C^n .

Si $\mathcal{A}'$ est l'ensemble des points M de $\mathcal{A}$ ainsi que $s(M_0, M) \in S$, alors S est l'ensemble des n -uples ordonnés $s(M_0, M)$ de C^n pour lesquelles M appartient à $\mathcal{A}'$.

Nous avons

$$s(M_0, M_0) = (0, \dots, 0) \in S \Rightarrow M_0 \in \mathcal{A}'$$

et donc $\mathcal{A}' \neq \emptyset$.

Soit $s: \mathcal{A}^2 \rightarrow C^n$ la surjection définie dans [1] et $s': \mathcal{A}'^2 \rightarrow C^n$ sa restriction à $\mathcal{A}'^2$.

Pour tout $(A, B) \in \mathcal{A}'^2$ nous avons $s'(A, B) = s(A, B) = s(A, M_0) + s(M_0, B) = -s(M_0, A) + s(M_0, B) \in S$ et donc $s': \mathcal{A}'^2 \rightarrow S$.

En supposant $\dim_C S = m > 0$ nous pouvons identifier les espaces S et C^m par isomorphisme.

Considérons le couple $(\mathcal{A}', C^m)$. La fonction s' satisfait les axiomes A_1 et A_2 de [1]; en effet, pour tout (A, B, C) de $\mathcal{A}'^3$ nous avons $s'(A, B) + s'(B, C) = s(A, B) + s(B, C) = s(A, C) = s'(A, C)$; la restriction de la fonction s' à $\{M_0\} \times \mathcal{A}'$ est évidemment une surjection (sur $S \equiv C^m$) et cette restriction est une injection car pour tous (M_0, M_1) et (M_0, M_2) de $\{M_0\} \times \mathcal{A}'$ nous avons

$$s'(M_0, M_1) = s'(M_0, M_2) \Rightarrow s(M_0, M_1) = s(M_0, M_2) \Rightarrow b(M_1) = b(M_2) \Rightarrow M_1 = M_2.$$

Donc le couple $(\mathcal{A}', C^m)$ est une espace affine complexe associé à l'espace vectoriel C^m .

Alors il existe la bijection $b': \mathcal{A}' \rightarrow C^m \equiv S$ donnée par $b'(M) = b(M_0, M)$ pour tout $M \in \mathcal{A}'$, et donc M_0 est l'origine de l'espace affine $(\mathcal{A}', C^m)$, qui est un sous-espace affine de l'espace affine $(\mathcal{A}, C^n)$ ou une variété linéaire

de dimension m de l'espace $(\mathcal{A}, C^n)$, qui passe par le point M_0 de $\mathcal{A}$ et a la direction $S \equiv C^m$.

3. La relation d'équipollence R_s définie sur $\mathcal{A}^2$ [1] induit sur A'^2 une relation d'équipollence R_s , donnée par

$$(A, B) R_s (D, C) \Leftrightarrow s'(A, B) = s'(D, C)$$

pour tous (A, B) et (D, C) dans $\mathcal{A}'^2$.

Soit $V_m(s') = \mathcal{A}'^2 / R_s$, l'ensemble des vecteurs libres associés à la variété linéaire $\mathcal{A}'$.

Pour tout $(A, B) \in \mathcal{A}'^2$, la classe d'équivalence $\overrightarrow{AB}$ modulo R_s inclut la classe d'équivalence $(\overrightarrow{AB})'$ modulo R_s .

La fonction $F' : V_m \rightarrow S \equiv C^m$ donnée par $F'((\overrightarrow{AB})') = s'(A, B) = b'(B) - b'(A)$ est une bijection ; à l'aide de cette bijection on peut définir sur V_m les lois

$$\overline{a}' + \overline{b}' = F'^{-1}(F'(\overline{a}') + F'(\overline{b}'));$$

$$\alpha \overline{a}' = F'^{-1}(\alpha F'(\overline{a}')), \quad \alpha \in C,$$

ainsi que l'ensemble V_m devient un espace vectoriel de dimension m sur C , isomorphe à $S \equiv C^m$ par l'isomorphisme F' .

Considérons le sousensemble V de l'espace V_n donné par les vecteurs libres $\overrightarrow{AB}$ de V_n pour lesquels il existe $(D, C) \in \mathcal{A}'^2$ ainsi que $(D, C) R_s (A, B)$.

L'ensemble V est un sousespace de dimension m de l'espace V_n , isomorphe à $S \equiv C^m$. En effet, si F_V est la restriction à V de l'isomorphisme

$F : V_n \rightarrow C^n$ [1], nous avons $F_V : V \rightarrow S \equiv C^m$ parce que $F_V(\overrightarrow{AB}) = F_V(\overrightarrow{DC}) = F(\overrightarrow{DC}) = F'(\overrightarrow{DC}) \in S$; donc pour tout $(\overline{u}, \overline{v}) \in V^2$ nous avons $F_V(\overline{u}) + F_V(\overline{v}) \in S$ c'est à-dire $\overline{u} + \overline{v} = F_V^{-1}(F_V(\overline{u}) + F_V(\overline{v})) \in V$ et pour tout $(\alpha, \overline{u}) \in C \times V$ nous avons $\alpha F_V(\overline{u}) \in S$ c'est-à-dire $\alpha \overline{u} = F_V^{-1}(\alpha F_V(\overline{u})) \in V$.

Les relations

$$F_V(\overline{u} + \overline{v}) = F_V(\overline{u}) + F_V(\overline{v}),$$

$$F_V(\alpha \overline{u}) = \alpha F_V(\overline{u})$$

montrent que F_V est un isomorphisme des espaces V et $S \equiv C^m$ d'où on déduit $\dim_C V = m$.

Du diagramme

$$\begin{array}{ccc} V_m & \xrightarrow{f} & V \\ F' \downarrow & & \downarrow F_V \\ & S \equiv C^m & \end{array}$$

il résulte que la fonction $f = F_V^{-1} \circ F'$ est un isomorphisme des espaces V_m et V , qui est donné par

$$f((\overrightarrow{AB})') = F_V^{-1}(F'((\overrightarrow{AB})')) = F_V^{-1}(s'(A, B)) = F_V^{-1}(s(A, B)) = F^{-1}(F(\overrightarrow{AB})) = \overrightarrow{AB}$$

et donc

$$f(\overrightarrow{AB})' = \overrightarrow{AB} \Leftrightarrow (\overrightarrow{AB})' = f^{-1}(\overrightarrow{AB}).$$

La fonction f^{-1} est un isomorphisme des espaces V et V_m d'où on obtient

$$(\bar{u} + \bar{v})' = \bar{u}' + \bar{v}', \quad (\alpha \bar{u})' = \alpha \bar{u}'.$$

En conclusion, les espaces vectoriels S , V_m et V sont isomorphes entre eux et ils sont des espaces de dimension m sur C .

On peut donc identifier les espaces S et V_m avec V

$$S \equiv V \Leftrightarrow s'(A, B) \equiv \overrightarrow{AB}, \quad V_m \equiv V \Leftrightarrow \bar{v}' \equiv \bar{v},$$

et on dit que la direction de la variété linéaire $\mathcal{A}'$ est V et que l'espace des vecteurs libres associés à $\mathcal{A}'$ est V .

On obtient alors que $\mathcal{A}'$ est l'ensemble des points M de $\mathcal{A}$ tels que $\overrightarrow{M_0 M} \in V$. Si $(\bar{a}_1, \dots, \bar{a}_m)$ est une base algébrique de l'espace V et si O est l'origine de l'espace affine $(\mathcal{A}, C^n)$ alors la variété linéaire $\mathcal{A}'$ est l'ensemble des points M de $\mathcal{A}$ pour lesquels il existe $(u_1, \dots, u_m) \in C^m$ ainsi que

$$\overrightarrow{OM} = u_1 \bar{a}_1 + \dots + u_m \bar{a}_m + \overrightarrow{OM}_0.$$

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ASUPRA SPAȚIULUI VECTORILOR LIBERI ASOCIAȚI UNEI VARIETĂȚI LINIARE A UNUI SPAȚIU AFIN

(Rezumat)

În această notă se continuă studiul din [1] prin punerea în evidență a structurii afine a varietăților liniare ale unui spațiu afîn.

et donc

$$f(\overline{AB}) = \overline{AB} \Leftrightarrow f(\overline{AB}) = \overline{f(A)f(B)}.$$

La fonction f est un isomorphisme des espaces N et N' et on obtient

$$(u+v)' = u' + v', \quad (au)' = au'.$$

En conclusion, les espaces vectoriels N , N' et N'' sont isomorphes entre eux et ils sont des espaces de dimension n sur C . On peut donc identifier les espaces N et N' avec N'' .

$$N \cong N' \Leftrightarrow N \cong N'', \quad N \cong N' \Leftrightarrow N \cong N''.$$

On voit que la direction de la variété linéaire $\mathcal{C}$ est N et que l'espace des vecteurs linéaires associés à $\mathcal{C}$ est N' .

On obtient alors que $\mathcal{C}$ est l'ensemble des points M de $\mathcal{C}$ tels que $M, M' \in N$. Si $(a_1, \dots, a_n)$ est une base algébrique de l'espace N et si θ est l'origine de l'espace affine $(\mathcal{C})$, alors la variété linéaire $\mathcal{C}$ est l'ensemble des points M de $\mathcal{C}$ pour lesquels il existe $(a_1, \dots, a_n) \in C^n$ ainsi que

$$\overline{OM} = na_1 + \dots + na_n + \overline{OM'}.$$

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ESPACE SPATIAL VECTORIEL LINÉAIRE ASSOCIÉ À UNE VARIÉTÉ LINÉAIRE À TOUT RELATIF ALGÈRE

(Reçu le 10/10/60)

Il est démontré que si $\mathcal{C}$ est une variété linéaire à tout relatif algèbre, alors l'espace des vecteurs linéaires associés à $\mathcal{C}$ est un espace vectoriel linéaire.

ASUPRA UNOR ALGEBRE REALE, ASOCIATIVE DE ORDIN TREI

DE

ANA CRĂCIUN

1. Ne propunem aici să determinăm toate tipurile de algebre de ordin trei, reale, comutative și asociative care să conțină ca subalgebră, algebra numerelor duale, dată prin

$$(1) \quad e_1 e_1 = e_1, \quad e_1 e_2 = e_2 e_1 = e_2, \quad e_2 e_2 = 0.$$

Fie e_1, e_2, e_3 o bază într-o algebră de ordin trei și, fără a micșora generalitatea, presupunem că a fost astfel aleasă, încât e_1, e_2 să constituie bază pentru algebra numerelor duale (1).

În felul acesta tabela de înmulțire este

	e_1	e_2	e_3
(2) e_1	e_1	e_2	$\lambda e_1 + \mu e_2 + \eta e_3$
e_2	e_2	0	$\alpha e_1 + \beta e_2 + \gamma e_3$
e_3	$\lambda e_1 + \mu e_2 + \eta e_3$	$\alpha e_1 + \beta e_2 + \gamma e_3$	$p e_1 + q e_2 + r e_3$

unde $\alpha, \beta, \gamma, \lambda, \mu, \eta, p, q, r$ sînt constante reale supuse condițiilor de asociativitate. Se observă că asociativitatea trebuie asigurată numai pentru produsele în care intervine elementul e_3 , întrucît elementele e_1, e_2 au produsul asociativ. Impunînd aceste condiții se obțin ecuațiile

$$(3) \quad \alpha = \gamma = 0, \quad \gamma \lambda = 0, \quad \gamma \mu = 0, \quad \gamma(\eta - 1) = 0, \quad \lambda \eta = 0, \quad \mu \eta = 0, \quad \eta(\eta - 1) = 0, \\ \lambda^2 + \eta p = p + r \lambda, \quad \lambda \mu + \mu \beta + \eta q = q + \mu r, \quad \beta^2 = p + r.$$

Relația $\eta(\eta - 1) = 0$ conduce la $\eta = 1$ și respectiv $\eta = 0$.

2. Fie $\eta = 1$. În acest caz condițiile (3) se reduc la $\lambda = \mu = \alpha = \gamma = 0$, $p = \beta^2 - r\beta$ cu β, r, q reali arbitrari. Fie familia de algebre dată de (4). În cazul $r = 2\beta$ idempotenții acestei familii rezultă :

$$x_1 = e_1, \quad x_2 = \frac{r - \beta}{r - 2\beta} e_1 - \frac{q}{(r - 2\beta)^2} e_2 - \frac{1}{r - 2\beta} e_3, \\ x_3 = -\frac{\beta}{r - 2\beta} e_1 + \frac{q}{(r - 2\beta)^2} e_2 + \frac{1}{r - 2\beta} e_3.$$

Efectuînd schimbarea de bază

$$u_1 = -\frac{\beta}{r-2\beta} e_1 + \frac{q}{(r-2\beta)^2} e_2 + \frac{1}{r-2\beta} e_3,$$

$$u_2 = \frac{r-\beta}{r-2\beta} e_1 - \frac{q}{(r-2\beta)^2} e_2 - \frac{1}{r-2\beta} e_3, \quad u_3 = e_2 \quad (r \neq 2\beta)$$

se obține (5), fapt care stabilește

Propoziția 1. *Toate algebrele (4), în cazul $r \neq 2\beta$, sînt izomorfe și ele se pot reprezenta, de exemplu, prin tabela (5).*

	e_1	e_2	e_3		u_1	u_2	u_3	
(4)	e_1	e_1	e_2	e_3	(5)	u_1	0	0
	e_2	e_2	0	βe_2		u_2	0	u_2
	e_3	e_3	βe_2	$(\beta^2 + r\beta) e_1 + qe_2 + re_3$		u_3	0	u_3

Se vede că algebra (5) este suma directă dintre algebra (1) și algebra $\mathbb{R}$ a numerelor reale, fiind în consecință polinomială, generată de un polinom de forma $\varphi(\lambda) = \lambda^2(\lambda - \lambda_0)$ [1]. Unitatea algebrei (5) este $u_1 + u_2$.

Dacă $r = 2\beta$, tabela (4) ia forma (6) care, cu schimbarea de bază $u_1 = e_1$, $u_2 = e_1 - \frac{1}{\beta} e_3$, $u_3 = \frac{q}{\beta^2} e_2$, ($\beta, q \neq 0$), conduce la (7). Algebra (7) este polinomială, avînd ca polinom generator $\varphi(\lambda) = \lambda^3$ [1]. Algebrele (5) și (7), deși polinomiale, sînt neizomorfe, așa cum rezultă din compararea polinoamelor lor generatoare [1]. Se constată de asemenea, că unitatea algebrei (1) este unitate și pentru algebra (7).

Dacă $q = 0$, $\beta \neq 0$, tabela (6) devine (6') care, cu schimbarea de bază $u_1 = e_1$, $u_2 = -e_1 + \frac{1}{\beta^2} e_2 + \frac{1}{\beta} e_3$, $u_3 = \frac{1}{\beta^2} e_2$ ($\beta \neq 0$) conduce la (8). Algebra (8) nu este polinomială, deoarece polinomul ei minimal este $\varphi(\lambda) = (\lambda - a)^2$, gradul lui fiind mai mic decît ordinul algebrei [1]. Prin calcul direct se constată că unitatea algebrei (1) este unitate și pentru algebra (8).

Algebra (8) nefiind polinomială nu este izomorfă cu respectiv algebrele (5) și (7).

Pentru $q \neq 0$, $\beta = 0$ din (6) obținem tabela (6'') care, cu schimbarea de bază $u_1 = e_1$, $u_2 = qe_2 + e_3$, $u_3 = qe_2$ ($q \neq 0$) conduce la o tabelă identică cu (7).

În sfîrșit, dacă $q = 0$, $\beta = 0$, din (6) obținem (6''') identică cu (8).

	e_1	e_2	e_3		u_1	u_2	u_3		
(5)	e_1	e_1	e_2	e_3	(7)	u_1	u_1	u_2	u_3
	e_2	e_2	0	βe_2		u_2	u_2	u_3	0
	e_3	e_3	βe_2	$-\beta^2 e_1 + q e_2 + 2 \beta e_3$		u_3	u_3	0	0

	e_1	e_2	e_3
(6')	e_1	e_1	e_2
	e_2	0	βe_2
	e_3	βe_2	$-\beta^2 e_1 + 2\beta e_3$

	u_1	u_2	u_3		e_1	e_2	e_3		e_1	e_2	e_3			
(8)	u_1	u_1	u_2	u_3	(6'')	e_1	e_1	e_2	e_3	(6''')	e_1	e_1	e_2	e_3
	u_2	u_3	0	0		e_2	e_2	0	0		e_2	e_2	0	0
	u_3	u_3	0	0		e_3	e_3	0	qe_3		e_3	e_3	0	0

Aceste rezultate dovedesc

Teorema 1. În cazul $\eta=1$ există trei tipuri de algebre neizomorfe de ordin trei, reale, asociative, comutative, care conțin ca subalgebră subalgebra numerelor duale; ele se pot reprezenta prin respectiv tabelele (5), (7), (8).

3. În cazul $\eta=0$, condițiile (3) de asociativitate devin: $\alpha=\gamma=0$, $q=\beta\mu-r\mu+\lambda\mu$, $p=\beta^2-r\beta$, $(\lambda-\beta)(r-\lambda-\beta)=0$. Ultima relație impune considerarea a două subcazuri:

1) Când $\lambda=\beta$, tabela (2) ia forma (9) cu β , μ , r reali, arbitrari. Idempotenții familiei (9), determinați în cazul $r \neq 0$, sînt $x_1=e_1$, $x_2=\frac{r-\beta}{r}e_1-\frac{\mu}{r}e_2+\frac{1}{r}e_3$, $x_3=-\frac{\beta}{r}e_1-\frac{\mu}{r}e_2+\frac{1}{r}e_3$, cu x_2 unitate. În (9) efectuînd schimbarea de bază $u_1=-\frac{\beta}{r}e_1-\frac{\mu}{r}e_2+\frac{1}{r}e_3$, $u_2=e_1$, $u_3=e_2$ ($r \neq 0$) se obține o tabelă identică cu (5), ceea ce arată că în acest caz toate algebrele familiei (9) sînt izomorfe cu algebra (5).

Dacă $r=0$, tabela (9) ia forma (10) care, cu schimbarea de bază $u_1=e_1$, $u_2=-\frac{2\mu}{\beta}e_2$, $u_3=e_1-\frac{\mu}{\beta}e_2-\frac{1}{\beta}e_3$ (μ , $\beta \neq 0$) conduce la (11).

		e_1	e_2	e_3	
(9)	e_1	e_1	e_2	$\beta e_1 + \mu e_2$	
	e_2	e_2	0	βe_2	
	e_3	$\beta e_1 + \mu e_2$	βe_2	$(\beta^2 - r\beta)e_1 + (2\beta\mu - \mu r)e_2 + re_3$	

		e_1	e_2	e_3		u_1	u_2	u_3
(10)	e_1	e_1	e_2	$\beta e_1 + \mu e_2$	(11)	u_1	u_1	u_2
	e_2	e_2	0	βe_2		u_2	u_2	0
	e_3	$\beta e_1 + \mu e_2$	βe_2	$\beta^2 e_1 + 2\beta\mu e_2$		u_3	u_3	0

Singurul idempotent al algebrei (11) este u_1 și, întrucît acesta nu este unitate, rezultă că tipul (11) nu este izomorf cu respectiv nici unul din tipurile (5), (7), (8).

Fie acum $\mu=0$, $\beta \neq 0$. În acest caz din (10) obținem tabela de înmulțire (10') care, cu schimbarea de bază $u_1=e_1$, $u_2=\frac{1}{\beta}e_2$, $u_3=e_1+\frac{1}{\beta}e_2-\frac{1}{\beta}e_3$ ($\beta \neq 0$), conduce la o tabelă identică cu (11).

Pentru $\beta=0$, $\mu \neq 0$, tabela (10) devine (10''), care cu schimbarea $u_1=e_1$, $u_2=\mu e_2$, $u_3=e_3$ ($\mu \neq 0$) se aduce la forma (11).

În sfîrșit, pentru a epuiza acest subcaz, să considerăm în (10) $\mu=\beta=0$. Găsim tabela de înmulțire (10''') care cu $u_1=e_1$, $u_2=e_2$, $u_3=e_2+e_3$ se aduce tot la forma (11).

	e_1	e_2	e_3		e_1	e_2	e_3		e_1	e_2	e_3			
(10')	e_1	e_1	e_2	βe_1	(10'')	e_1	e_1	e_2	μe_2	(10''')	e_1	e_1	e_2	0
	e_2	e_2	0	βe_2		e_2	e_2	0	0		e_2	e_2	0	0
	e_3	βe_1	βe_2	$\beta^2 e_1$		e_3	μe_2	0	0		e_3	0	0	0

Am stabilit astfel

Propoziția 2. În cazul $\eta=0$, $\lambda=\beta$, familia (2) conține numai două tipuri de algebre neizomorfe și acestea se pot reprezenta prin (5) și (11).

2) Fie $\lambda=r-\beta$. În acest caz tabela (2) ia forma (12), în care efectuând schimbarea de bază $u_1=e_1$, $u_2=\frac{\mu}{r}e_2$, $u_3=\frac{\beta}{r}e_1+\frac{1}{r}e_3$ ($\mu, r \neq 0$) și notînd $\beta/r=k$, se obține tabela (13).

	e_1	e_2	e_3		u_1	u_2	u_3	
(12)	e_1	e_1	e_2	$(r-\beta)e_1+\eta e_2$	(13)	u_1	u_2	u_1+u_2
	e_2	e_2	0	βe_2		u_2	0	$2ku_2$
	e_3	$(r-\beta)e_1+\mu e_2$	βe_2	$(\beta^2-r\beta)e_1+re_3$		u_3	u_1+u_2	$2ku_2+u_3$

Fie k_1 și k_2 două valori date lui k și, să notăm cu φ o funcție care reprezintă izomorf algebra (13) corespunzătoare valorii k_1 , pe cea corespunzătoare valorii k_2 . Deci

$$\varphi(u_1)=Au_1+B u_2+C u_3, \quad \varphi(u_2)=Du_1+E u_2+F u_3, \quad \varphi(u_3)=Gu_1+H u_2+I u_3.$$

Condiția $\varphi^2(u_1)=\varphi(u_1)$ conduce la

$$\varphi(u_1)=\frac{2k_1}{1-4k_1}u_2+u_3, \quad \varphi(u_1)=-u_1+\frac{2(1-k_1)}{4k_1-3}u_2+u_3, \quad \varphi(u_1)=u_1,$$

iar condiția $\varphi^2(u_2)=0$ atrage $\varphi(u_2)=Eu_2$.

$$\text{Dacă } \varphi(u_1)=\frac{2k_1}{1-4k_1}u_2+u_3, \quad \varphi(u_2)=Eu_2, \quad \varphi(u_3)=Gu_1+Hu_2+Iu_3,$$

condiția $\varphi(u_1)\varphi(u_2)=\varphi(u_2)$ conduce la $E=0$, ceea ce este imposibil. La aceeași concluzie se ajunge și în cazul

$$\varphi(u_1)=-u_1+\frac{2(1-k_1)}{4k_1-3}u_2+u_3, \quad \varphi(u_2)=Eu_2, \quad \varphi(u_3)=Gu_1+Hu_2+Iu_3.$$

Fie deci $\varphi(u_1)=u_1$, $\varphi(u_2)=Eu_2$, $\varphi(u_3)=Gu_1+Hu_2+Iu_3$. Condițiile $\varphi(u_1)\varphi(u_3)=\varphi(u_1)+\varphi(u_2)$ și $\varphi(u_2)\varphi(u_3)=2k_2\varphi(u_2)$ conduc pentru $\varphi(u_3)$ la forma

$$\varphi(u_3)=\frac{2(k_1-k_2)}{2k_1-1}u_1+\left(\frac{1-2k_2}{2k_1-1}+E\right)u_2+\frac{2k_2-1}{2k_1-1}u_3.$$

Condiția $\varphi^2(u_3)=2k_2\varphi(u_2)+\varphi(u_3)$ implică pentru k_1 și k_2 relația $(2k_2-1)^2/(2k_1-1)^2=(2k_2-1)/(2k_1-1)$ care nu poate avea loc decît dacă $k_2=1/2$ sau $k_1=k_2$. Valoarea $k_2=1/2$ nu este posibilă întrucît $\varphi(u_1)$, $\varphi(u_2)$, $\varphi(u_3)$ nu mai pot forma o bază. Singurul caz posibil rămîne $k_1=k_2$, ceea ce conduce la izomorfismul identic, stabilind astfel

Teorema 2. *Toate algebrele (13) sînt neizomorfe. Idempotenții familiei de algebre (13) sînt, în cazul $k \neq 1/4$, $k \neq 3/4$,*

$$x_1 = u_1, \quad x_2 = \frac{2k}{1-4k} u_2 + u_3, \quad x_3 = -u_1 + \frac{2(k-1)}{3-4k} u_2 + u_3.$$

Faptul că algebra (11) admite numai un singur idempotent, arată că ea nu este izomorfă cu (13). Familia de algebre (13) nu are unitate, ceea ce demonstrează neizomorfismul ei cu algebrele (5), (7), (8).

Au rămas necercetate tipurile obținute din (13) pentru valorile lui k egale cu $1/4$ și respectiv $3/4$. Pentru aceste valori ale lui k , din (13) se obțin algebrele

		u_1	u_2	u_3			u_1	u_2	u_3
(14)		u_1	u_2	$u_1 + u_2$	(15)		u_1	u_3	$u_1 + u_2$
	u_2	u_2	0	$\frac{1}{2} u_2$		u_2	u_2	0	$\frac{3}{2} u_2$
	u_3	$u_1 + u_2$	$\frac{1}{2} u_2$	$\frac{1}{2} u_2 + u_3$		u_3	$u_1 + u_2$	$\frac{3}{2} u_2$	$\frac{3}{2} u_2 + u_3$

evident neizomorfe între ele, întrucît corespund la valori diferite ale lui k .

Idempotenții calculați în (14) sînt $x_1 = u_1$, $x_2 = -u_1 - \frac{3}{4} u_2 + u_3$, se constată

că algebra (14) nu are unitate. Idempotenții din (15) sînt $x_1 = u_1$, $x_2 = -\frac{3}{4} u_2 + u_3$ și, de asemenea, această algebra nu are unitate. Acestea demonstrează că algebrele (14) și (15) nu sînt izomorfe respectiv cu nici una din algebrele (5), (7), (8) și (11).

Pentru $\mu \neq 0$, $r = 0$, algebra (12) devine (12'), care, apoi, cu transformarea de bază $u_1 = e_1$, $u_2 = \frac{\mu}{\beta} e_2$, $u_3 = \frac{1}{\beta} e_3$ ia forma (16),

		e_1	e_2	e_3			u_1	u_2	u_3
(12')		e_1	e_2	$-\beta e_1 + \mu e_2$	(16)		u_1	u_2	$-u_1 + u_2$
	e_2	e_2	0	βe_2		u_2	u_2	0	u_2
	e_3	$-\beta e_1 + \mu e_2$	βe_2	$\beta^2 e_1$		u_3	$-u_1 + u_2$	u_2	u_1

Algebra (16) admite ca unic idempotent pe $x = u_1$, care nu este unitate pentru algebra, deci ea nu este izomorfă cu respectiv nici una din algebrele (5), (7), (8), (13), (14) și (15). Nilpotenții de indice doi calculați pentru algebrele (11) și (16) au forma $\{bu_2 + cu_3\}$, b, c arbitrari și respectiv $\{\alpha(-2u_1 + u_2 - 2u_3)\}$ și $\{\beta u_2\}$, α, β arbitrari. Prin calcul direct se constată că în timp ce produsul oricărui două asemenea elemente din (11) este zero, proprietatea nu are loc pentru algebra (16), ceea ce arată că algebrele (11) și (16) sînt neizomorfe.

Fie acum în (12) $\mu = 0$, $r \neq 0$. Obținem tabela (17) care, cu schimbarea de bază $u_1 = e_1$, $u_2 = e_2$, $u_3 = \frac{\beta}{r} e_1 + \frac{1}{r} e_3$, și cu notația $2\beta/r = k$ ia forma (17')

(17)	<table style="border-collapse: collapse;"> <tr> <th></th> <th>e_1</th> <th>e_2</th> <th>e_3</th> </tr> <tr> <th>e_1</th> <td>e_1</td> <td>e_2</td> <td>$(r-\beta)e_1$</td> </tr> <tr> <th>e_2</th> <td>e_2</td> <td>0</td> <td>βe_2</td> </tr> <tr> <th>e_3</th> <td>$(r-\beta)e_1$</td> <td>βe_2</td> <td>$(\beta^2-r\beta)e_1+re_3$</td> </tr> </table>		e_1	e_2	e_3	e_1	e_1	e_2	$(r-\beta)e_1$	e_2	e_2	0	βe_2	e_3	$(r-\beta)e_1$	βe_2	$(\beta^2-r\beta)e_1+re_3$	(17')	<table style="border-collapse: collapse;"> <tr> <th></th> <th>u_1</th> <th>u_2</th> <th>u_3</th> </tr> <tr> <th>u_1</th> <td>u_1</td> <td>u_2</td> <td>u_1</td> </tr> <tr> <th>u_2</th> <td>u_2</td> <td>0</td> <td>ku_2</td> </tr> <tr> <th>u_3</th> <td>u_1</td> <td>ku_2</td> <td>u_3</td> </tr> </table>		u_1	u_2	u_3	u_1	u_1	u_2	u_1	u_2	u_2	0	ku_2	u_3	u_1	ku_2	u_3
	e_1	e_2	e_3																																
e_1	e_1	e_2	$(r-\beta)e_1$																																
e_2	e_2	0	βe_2																																
e_3	$(r-\beta)e_1$	βe_2	$(\beta^2-r\beta)e_1+re_3$																																
	u_1	u_2	u_3																																
u_1	u_1	u_2	u_1																																
u_2	u_2	0	ku_2																																
u_3	u_1	ku_2	u_3																																

Fie k_1 și k_2 două valori arbitrare date lui k și, din nou, φ o funcție ce reprezintă izomorf algebra corespunzătoare valorii k_1 pe algebra corespunzătoare valorii k_2 . Fie deci $\varphi(u_1)=Au_1+Bu_2+Cu_3$, $\varphi(u_2)=Du_1+Eu_2+Fu_3$, $\varphi(u_3)=Gu_1+Hu_2+Iu_3$. Condițiile $\varphi^2(u_1)=\varphi(u_1)$, $\varphi^2(u_2)=0$, $\varphi^2(u_3)=\varphi(u_3)$ implică considerarea a trei cazuri și anume:

a) $\varphi(u_1)=u_1$, $\varphi(u_2)=Eu_2$, $\varphi(u_3)=-u_1+u_3$. Condiția $\varphi(u_1)\cdot\varphi(u_3)=\varphi(u_1)$ conduce la $u_1=0$, care nu este posibil.

b) $\varphi(u_1)=-u_1+u_3$, $\varphi(u_2)=Eu_2$, $\varphi(u_3)=u_3$. Condiția $\varphi(u_1)\cdot\varphi(u_2)=\varphi(u_2)$ conduce la $E=0$, care de asemenea nu este posibil.

c) $\varphi(u_1)=u_1$, $\varphi(u_2)=Eu_2$, $\varphi(u_3)=u_3$. Condiția $\varphi(u_2)\cdot\varphi(u_3)=k_2\varphi(u_2)$ conduce la $k_1=k_2$, deci la izomorfismul identic, ceea ce stabilește

Teorema 3. *Toate algebrele (17) sînt neizomorfe.*

Pentru familia de algebre (17) găsim idempotenții $x_1=e_1$, $x_2=\frac{\beta}{r}e_1+\frac{1}{r}e_3$, $x_3=\frac{\beta-r}{r}e_1+\frac{1}{r}e_3$, nici unul dintre ei nefiind unitate, deci algebra (17) nu este izomorfă cu nici una dintre algebrele (5), (7), (8), (11), (14), (15), (16). De asemenea se constată că între idempotenții familiei de algebre (17) există relația $x_2-x_3=x_1$ care însă nu are loc pentru idempotenții, considerați pentru $k\neq 1/2$, ai algebrilor (13). Aceasta stabilește

Teorema 4. *Nici o algebră din familia (17) nu este izomorfă cu respectiv nici o algebră din familia (13), unde $k\neq 1/2$. Algebra pentru $k=1/2$ din (13) se corespunde izomorf cu algebra pentru $k=1$ din (17).*

Dacă $\mu=0$, $r=0$, din (12) găsim tabela (12') în care efectuînd schimbarea $u_1=e_1$, $u_2=\frac{1}{\beta}e_2$, $u_3=\frac{1}{\beta}e_3$ ($\beta\neq 0$) se obține algebra dată prin (18).

(12')	<table style="border-collapse: collapse;"> <tr> <th></th> <th>e_1</th> <th>e_2</th> <th>e_3</th> </tr> <tr> <th>e_1</th> <td>e_1</td> <td>e_2</td> <td>$-\beta e_1$</td> </tr> <tr> <th>e_2</th> <td>e_2</td> <td>0</td> <td>βe_2</td> </tr> <tr> <th>e_3</th> <td>$-\beta e_1$</td> <td>βe_2</td> <td>$\beta^2 e_1$</td> </tr> </table>		e_1	e_2	e_3	e_1	e_1	e_2	$-\beta e_1$	e_2	e_2	0	βe_2	e_3	$-\beta e_1$	βe_2	$\beta^2 e_1$	(18)	<table style="border-collapse: collapse;"> <tr> <th></th> <th>u_1</th> <th>u_2</th> <th>u_3</th> </tr> <tr> <th>u_1</th> <td>u_1</td> <td>u_2</td> <td>$-u_1$</td> </tr> <tr> <th>u_2</th> <td>u_2</td> <td>0</td> <td>u_2</td> </tr> <tr> <th>u_3</th> <td>$-u_1$</td> <td>u_2</td> <td>u_1</td> </tr> </table>		u_1	u_2	u_3	u_1	u_1	u_2	$-u_1$	u_2	u_2	0	u_2	u_3	$-u_1$	u_2	u_1
	e_1	e_2	e_3																																
e_1	e_1	e_2	$-\beta e_1$																																
e_2	e_2	0	βe_2																																
e_3	$-\beta e_1$	βe_2	$\beta^2 e_1$																																
	u_1	u_2	u_3																																
u_1	u_1	u_2	$-u_1$																																
u_2	u_2	0	u_2																																
u_3	$-u_1$	u_2	u_1																																

Algebra (18) are un singur idempotent u_1 , care însă nu este unitate. Deci (18) nu este izomorfă cu respectiv algebrele (5), (7), (8), (13), (14), (15), (17). Se constată că algebra (18) admite două familii de nilpotenți de indice doi, anume $\{a(u_1+u_3)\}$ și $\{bu_2\}$, a, b arbitrari. În timp ce produsul oricărui doi nilpotenți de indice doi din (11) este zero, proprietatea nu are loc pentru (18), ceea ce arată că algebrele (11) și (18) nu sînt izomorfe. În sfîrșit, dacă algebrele (18) și (16) ar fi izomorfe ar trebui să existe o funcție φ care să reprezinte izomorf cele două algebre. Fie deci $\varphi(u_1)=Au_1+Bu_2+Cu_3$, $\varphi(u_2)=Du_1+Eu_2+Fu_3$, $\varphi(u_3)=Mu_1+Nu_2+Pu_3$. Condițiile $\varphi^2(u_1)=\varphi(u_1)$, $\varphi^2(u_2)=0$, $\varphi(u_1)\cdot\varphi(u_2)=\varphi(u_2)$ conduc la $\varphi(u_1)=u_1$, $\varphi(u_2)=Eu_2$. Condițiile

$\varphi(u_1) \cdot \varphi(u_3) = -\varphi(u_1)$ și $\varphi(u_2) \cdot \varphi(u_3) = \varphi(u_2)$ dau lui $\varphi(u_3)$ forma $\varphi(u_3) = -u_2 + u_3$, pentru care $\varphi^2(u_3) \neq \varphi(u_1)$. Deci algebrele (16) și (18) sînt neizomorfe.

Dacă în (12) $\mu = r = \beta = 0$ și se face schimbarea de bază $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_2 + e_3$, se regăsește algebra (11).

4. Rezultatele găsite permit următoarele observații:

1. Există trei tipuri de algebre reale neizomorfe de ordin trei, asociative și cu unitate care să conțină ca subalgebră, algebra numerelor duale și acestea se pot reprezenta prin respectiv tabelele (5), (7) și (8).

2. Unitatea algebrei numerelor duale coincide cu unitatea algebrei de ordin trei numai în algebrele reprezentate prin tabelele (7) și (8).

3. Există două familii neizomorfe de algebre de ordin trei, reale, asociative, comutative, fără unitate care să conțină ca subalgebră algebra (1), fiecare din ele conținînd respectiv cîte o infinitate de algebre neizomorfe. Acestea s-au reprezentat prin tabelele (13) și (17). În plus, tot fără unitate, neizomorfe între ele și neizomorfe cu familiile de algebre (13) și (17) sînt algebrele date de respectiv (11), (16) și (18).

4. Algebra cu tabela (5) este decompozabilă în suma directă dintre $\mathbb{R}$ și algebra (1). Algebrele reprezentate prin tabelele (7) și (8) sînt indecompozabile pentru că admit ca unic idempotent unitatea și deci se pot reprezenta ca sume nedirecte dintre $\mathbb{R}$ și respectiv radicalul fiecărei algebre, [2].

5. Se constată că algebra (1) este ideal maximal pentru algebrele date prin tabelele (11), (13), (16), (17) și (18).

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SUR CERTAINES ALGÈBRES RÉELLES ASSOCIATIVES D'ORDRE TROIS

(Résumé)

On détermine les algèbres réelles d'ordre trois, associatives et commutatives, qui contiennent comme sousalgèbre, l'algèbre des nombres duaux, donnée par (1). On démontre qu'il existe trois types d'algèbres avec unité, donnés par (5), (7) et (8), et deux familles non-isomorphes (13) et (17), qui contiennent séparément une infinité d'algèbres non-isomorphes sans unité. En outre, on a trouvé encore trois types, aussi sans unité, qui sont représentés par les tables (11), (16) et (18).

$\varphi(w_1) = -\varphi(w_2) = \varphi(w_3) = \varphi(w_4)$ și $\varphi(w_5) = \varphi(w_6)$ dan în $\varphi(w_7)$ forma $\varphi(w_8) = -w_7$, pentru care $\varphi(w_9) = \varphi(w_{10})$. Deci algebra (10) și (11) sînt neisomorfe.

Deci în (12) $x = y = z = 0$ și se face schimbarea de variabile $w_1 = x, w_2 = y, w_3 = z, w_4 = x + y + z$ și se obține algebra (11).

4. Rezultatele găsite permit următoarele observații:

1. Există trei tipuri de algebre reale neisomorfe de ordin trei, asociative și cu unitate care să conțină ca subalgebră, algebra numerelor duali și acestea se pot reprezenta prin respective tabelele (5), (7) și (8).

2. Unitatea algebrilor numerelor duali coincide cu unitatea algebrei de ordin trei numai în algebrele reprezentate prin tabelele (7) și (8).

3. Există două familii neisomorfe de algebre de ordin trei, reale, asociative, comutative, 177 unitate care să conțină ca subalgebră algebre (1). Fiecare din ele conținând respectiv câte o infinitate de algebre neisomorfe. Acestea s-au reprezentat prin tabelele (13) și (17). În plus, tot atât unitate, neisomorfe între ele și neisomorfe cu familiile de algebre (13) și (17) sînt algebre date de respectiv (11), (10) și (12).

4. Algebra cu tabela (5) este decompozabilă în suma directă dintre $\mathbb{R}^2$ și algebra (4). Algebrele reprezentate prin tabelele (7) și (8) sînt indecompozabile pentru că admit ca unic idempotent unitatea și deci se pot reprezenta ca sume nedirecte dintre $\mathbb{R}^2$ și respectiv radicalul înălțării algebre (7).

5. Se constată că algebra (1) este ideal maximal pentru algebrele date prin tabelele (11), (13), (10), (17) și (18).

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Cluj-Napoca

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LES ALGÈBRES RÉELLES ASSOCIATIVES D'ORDRE TROIS

(Reçu)

On détermine les algèbres réelles d'ordre trois, associatives et commutatives, qui contiennent comme sous-algèbre l'algèbre des nombres duals, donnée par (1). On démontre qu'il existe trois types d'algèbres avec unité, données par (2), (3) et (4), et deux familles non-isomorphes (13) et (17), qui contiennent séparément une infinité d'algèbres non-isomorphes sans unité. En outre, on trouve encore trois types, aussi sans unité, qui sont représentés par les tables (11), (10) et (12).

QUASI-VARIATIONAL INEQUALITIES WITH NON-MONOTONE IMPLICIT RESTRICTION

BY

ROBERT T. VESCAN

1. Consider a reflexive ordered Banach space V with dual V' and a functional $f \in V'$. Suppose that V is dense in a Banach space E and the injection $V \hookrightarrow E$ is linear compact. Let $a(\cdot, \cdot)$ be a coercive bilinear continuous form on $V \times V$ and K_u subsets of V depending on $u \in V$. If one looks for an element $u \in V$ so that $u \in K_u$ and $a(u, w-u) \geq f(w-u)$, $\forall w \in V$ with $w \in K_u$, then such a problem is called a quasi-variational inequality (QVI). This generalization of classical variational inequalities (see [6]) is due to A. Bensoussan and J. L. Lions (see e.g. [2]).

Other formulations of implicit variational inequalities, entailing terms which depend on the possible solution, have been studied by many authors: [7], [4], [1], [3], [5] etc.

The existence of solutions for the concrete QVI enunciated just before were established under different growth assumptions on K_u (as depending on $u \in V$) in [2] and [7]. In [9], besides a general existence theorem, we proved a result applied to parabolic QVI which we restate here in some stationary cases.

In what follows (see also the Remarks 1 below) we assume weak continuity and a nonvoid intersection property concerning the sets K_u (hypotheses (ii) and (iii) in Theorem 1) and we deduce the existence of solutions by means of a fixed point theorem applied to the usual selection map (see [4]) of the QVI.

Theorem 1. Let $V, V', f, E, a(\cdot, \cdot)$ be as above. Suppose that

$$|f(v)| \leq A\|v\|, \quad |a(u, v)| \leq B\|u\|\|v\| \quad \text{and} \quad a(v, v) \geq C\|v\|^2$$

for all $u, v \in V$, where A, B, C are fixed strictly positive constants and $\|\cdot\|$ is the norm in V . For $w \in V$, denote by S_w the ball $\{u \in V; \|u\| \leq k_w\}$ with

$$k_w = \frac{1}{2C} \{B\|w\| + A + [(B\|w\| + A)^2 + 4CA\|w\|]^{\frac{1}{2}}\}.$$

Let us consider further an operator $M: V \rightarrow V$. If

- (i) the positive cone P is closed in V ;
- (ii) $\forall w \in V$, S_w is continuous on S_w from the norm topology of E to the weak topology of V ;
- (iii) there exists $w_0 \in V$ such that M is bounded from below on S_{w_0} , namely $w_0 \leq Mu$, $\forall u \in S_{w_0}$; then the

(QVI) $a(u, w-u) \geq f(w-u)$, $\forall w \in V$ with $w \leq Mu$ has at least a solution $u \in V$ with $u \leq Mu$.

Remark 1. Let us set of f that E is not ordered and we have no latticial assumptions as in [2] or [7]. The monotonicity hypotheses (1.8) from [2] or (11) from [7] and the strong continuity (1.7) from [2] are replaced by (ii) and (iii). One can possibly verify that in this situation all conditions of the general existence Theorem 4.1 from [4] are fulfilled, but it is more convenient to give a direct proof as below.

Proof of Theorem 1. Let us define the selection map of the QVI, $T: S_{w_0} \rightarrow V$ by $v = T(u)$, for u fixed in S_{w_0} , if and only if v is the solution of the variational problem

$$\begin{cases} v \leq Mu, \\ a(v, w-v) \geq f(w-v), \quad \forall w \in V \text{ with } w \leq Mu. \end{cases}$$

By the closeness of the cone P in V , it follows that the set $K_u = \{w \in V; w \leq Mu\}$ is closed convex in V . Then one can appeal to a well known existence theorem of G. Stampacchia — J. L. Lions (see [6; §2]) to claim that $T(u)$ is correctly defined. Since $a(v, w_0-v) - f(w_0-v) \leq B\|v\| \|w_0\| - C\|v\|^2 + A\|w_0\| + A\|v\|$, one sees that $a(v, w_0-v) < f(w_0-v)$ for all v with $\|v\| > k_{w_0}$, that is all $v \in V \setminus S_{w_0}$. But by hypothesis (iii) $w_0 \in K_u$ for all $u \in S_{w_0}$, therefore we must have $v = T(u) \in S_{w_0}$, $\forall u \in S_{w_0}$. Thus $T(S_{w_0}) \subset S_{w_0}$ and the ball S_{w_0} is clearly weakly compact in V .

We show next that T is weakly continuous. Due to the compactness of the injection $V \rightarrow E$, S_{w_0} is also relatively compact in the Banach space E . As we have already remarked in our work [8] the weak topology of V and the norm topology of E are compatible on V , moreover they are equivalent on S_{w_0} (see also [9]).

If (u_n) is a sequence in S_{w_0} which converges weakly in V to u , then $v_n = T(u_n) \in S_{w_0}$ converges weakly in V to $v \in S_{w_0}$. As noticed before (u_n) has equally the limit u in the topology of E .

$a(\cdot, w - \cdot)$ is weakly upper semi-continuous on V , $f(w - \cdot)$ is weakly continuous on V , M satisfies hypothesis (ii) and the cone P is closed in V , so that the inequalities

$$a(v_n, w - v_n) \geq f(w - v_n), \quad \forall w \in V \text{ with } w \leq Mu_n,$$

imply

$$a(v, w - v) \geq f(w - v), \quad \forall w \in V \text{ with } w \leq Mu.$$

Similarly, the inequalities $v_n \leq Mu_n$ imply $v \leq Mu$. This conclusions mean that $v = T(u)$.

Thus by Schauder-Tihonov's Theorem $T: S_{w_0} \rightarrow S_{w_0}$ must have a fixed point $u = T(u)$, i.e. an $u \in S_{w_0}$ which is a solution of the QVI as desired in the statement of our theorem.

2. Let D be a bounded open subset with smooth boundary in the n -dimensional Euclidean space R^n and consider $V = H^1(D)$ the usual Sobolev space. Let us take

$$a(u, v) = \int_D \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + uv \right) dx,$$

just the inner product in $H^1(D)$. Further, identify the Hilbert space $L^2(D)$ with its dual and put $E=L^2(D)$; thus $V \subset E \subset V'$ and if f, g belong to E , then $f, g \in V'$ and $f(v) = \int_D f v dx$, $g(v) = \int_D g v dx$ for all $v \in H^1(D)$. Define $M: H^1(D) \rightarrow H^1(D)$ by

$$Mv = a + \int_D g v dx, \quad \forall v \in H^1(D),$$

with $a \in R$ constant and g fixed in $L^2(D)$.

We consider a problem involving the operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and the implicit restriction M , written formally in the distribution sense as

$$\left. \begin{aligned} u &\leq Mu \\ -\Delta u + u - f &\leq 0 \\ (u - Mu)(-\Delta u + u - f) &= 0 \end{aligned} \right\} \text{ in } D$$

with the boundary condition

$$u \leq Mu, \quad \frac{\partial u}{\partial \nu} \leq 0, \quad (u - Mu) \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D,$$

$\left(\frac{\partial u}{\partial \nu}\right)$ is the normal derivative of u on ∂D associated with Δ . By a usual method, this problem has a weak formulation as a QVI which is broached in

Theorem 2. Consider D , $a(\cdot, \cdot)$, f, g, M as above and denote $A = \|f\|_{L^2(D)}$, $G = \|g\|_{L^2(D)}$ and $m = (\text{meas } D)^{1/2}$. If the system of inequalities

$$\begin{cases} (2 - Gm)t \geq GA - 2a, \\ (1 - Gm)t^2 + (2a - Gma - GA - G^2Am)t + a^2 - aGA \geq 0, \\ t \geq 0 \end{cases}$$

admits at least a solution $t \in R$, then the QVI

$$\begin{cases} u \leq Mu \text{ in } H^1(D) \\ a(u, w - u) = \langle u, w - u \rangle_{H^1(D)} \geq \int_D f \cdot (w - u) dx, \quad \forall w \in H^1(D) \end{cases}$$

with $w \leq Mu$ with the non-monotone restriction M has a solution $u \in H^1(D)$.

Remark 2. It is to be noticed that M is monotone only if g is constant on D . It is not difficult to see that the given real inequalities have a common solution $t \in R$, in case e.g.

$$1) \quad 1 - GM > 0 \Leftrightarrow \|g\|_{L^2} < \frac{1}{(\text{meas } D)^{1/2}},$$

or

$$2) \quad 1 - GM = 0 \text{ and } a \geq GA \Leftrightarrow \|g\|_{L^2} = \frac{1}{(\text{meas } D)^{1/2}} \text{ and } a \geq \|g\|_{L^2} \|f\|_{L^2} \text{ etc.}$$

Proof of Theorem 2. All the hypotheses of Theorem 1 on V , V' , f , E , $a(\cdot, \cdot)$, M are satisfied with

$A = \|f\|_{L^1(D)}$, $B = C = 1$ and $P = \{v \in H^1(D) ; v \geq 0 \text{ a.e. on } D\}$.

Moreover, if $v_n \in L^2(D)$ and v_n converges in $L^2(D)$ to v , then Mv_n, Mv are constant functions and clearly $Mv_n \rightarrow Mv$ in $H^1(D)$. Concerning (iii) we note that we must have simultaneously

$$w_0 \leq a + \int_D gu \, dx, \quad \forall u \text{ with } \|u\|_{H^1} \leq k_w,$$

and

$$k_w = \frac{1}{2} [\|w_0\|_{H^1} + A + (\|w_0\|_{H^1}^2 + GA\|w_0\|_{H^1} + A^2)^{1/2}].$$

The first inequalities are verified if

$$w_0(x) \leq a - \|g\|_{L^1} k_w \leq a - \left| \int_D gu \, dx \right| \quad \text{a.e. on } D.$$

Thus condition (iii) can be positively tested by a constant negative function $w_0(x) \equiv -t \leq 0$ with

$$\|w_0\|_{H^1} = tm \quad \text{if} \quad -t \leq a - G \frac{1}{2} [tm + A + (t^2 m^2 + GAtm + A^2)^{1/2}].$$

Consequently one obtains the system just given in the statement of Theorem 2.

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INEGALITĂȚI CVASI-VARIAȚIONALE CU RESTRICȚII IMPLICITE NEMONOTONE

(Rezumat)

În cadrul problemei existenței soluțiilor pentru probleme concrete de inegalități cvasi-variaționale, în condiții de continuitate slabă (ipotezele (ii) și (iii) din teorema 1), se demonstrează existența soluțiilor pe baza unei teoreme de punct fix.

SOME VERTICAL AND HORIZONTAL CONVEX FUNCTIONS ON THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

BY

FULGA POP

0. Introduction. In a previous paper [1] we defined the notion of convex function with respect to a distribution on a Riemannian manifold and we studied the convexity of the vertical lift of a smooth function to the tangent bundle.

In this Note we consider the vertical and horizontal convexity of a Pfaff form of a Riemannian manifold M , considered as a function on TM . In particular we investigate the vertical and horizontal convexity of the complete lift of a smooth function to the tangent bundle.

1. Review of Some Definitions and Results on Convex Function with respect to a distribution on a Riemannian manifold [1] and on the tangent bundle [3], [4]. Let M be a Riemannian manifold with respect to the metric g and the Levi-Civita connection ∇ , having the local expressions g_{ij} and Γ_{ij}^k respectively. Let $D = D(M)$ be a distribution on the manifold M (a subbundle of the tangent bundle TM).

Definition 1.1. A smooth function $f: M \rightarrow \mathbb{R}$ is called a D -convex function if for every point $x \in M$ and for every vector field $X \in D$, we have

$$(1) \quad \text{Hess } (f)(X, X)_x = \langle \nabla_X df, X \rangle_x = X(df(x)) - df(\nabla_X X) \geq 0.$$

If the function f is TM -convex function then it is a convex function in the sense of [2]. If $-f$ is a D -convex function, then we say that f is a D -concave function.

In local coordinates the condition (1) is

$$(2) \quad \sum_{i,j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^h \frac{\partial f}{\partial x^h} \right) X^i X^j \geq 0, \text{ for every } X = \sum_i X^i \frac{\partial}{\partial x^i} \in D(M).$$

Proposition 1.2. Let D be a distribution on a Riemannian manifold and let I be an integral manifold of D . If $f: M \rightarrow \mathbb{R}$ is a D -convex function, then the restriction $f' = f|I: I \rightarrow \mathbb{R}$ is a convex function on I .

Proposition 1.3. Let (M, D) be a Riemannian foliated manifold and let $f: M \rightarrow \mathbb{R}$ be a smooth function. Then, f is a D -convex (a foliated

convex function) if and only if the restriction of the function f to every leaf of the foliation D is a convex function.

If M is a Riemannian C^∞ -differentiable n -dimensional manifold, let TM be the tangent bundle with the natural projection $\pi: TM \rightarrow M$. We use notations and results on the tangent bundle from [3] and [4]. If (U, x^i) is a system of coordinate neighborhoods of M , with (x^i) a system of local coordinates defined in the neighborhood U , then $(\pi^{-1}(U), x^i, y^i)$ is a system of coordinate neighborhoods for TM , if (y^i) are the coordinates of a vector of $T_x M$, $x \in U$ in the natural frame $\partial/\partial x^1, \dots, \partial/\partial x^n$.

Consider the tangent bundle TM as a Riemannian manifold with the Riemannian metric G of Sasaki [3] (the metric I+III, [4]). Let $\tilde{\nabla}$ be the Levi-Civita connection of G and let V and H be, respectively, the vertical and horizontal distributions on the Riemannian manifold TM .

2. The Convexity of the Pfaff Forms on a Riemannian Manifold considered as functions on the tangent bundle. Denote by $\mathcal{F}_s^r(M)$ the set of all tensor fields of class C^∞ and of type (r, s) in M . Let $\iota: \mathcal{F}_s^0(M) \rightarrow \mathcal{F}_{s-1}^0(M)$ the mapping defined in local coordinates by

$$(3) \quad \iota(\sum_{i_1, \dots, i_s} dx^{i_1} \otimes \dots \otimes dx^{i_s}) = \sum_{i_1, \dots, i_s} y^{i_1} dx^{i_2} \otimes \dots \otimes dx^{i_s}.$$

In particular, if we take $\iota: \mathcal{F}_1^0(M) \rightarrow \mathcal{F}_0^0(M)$, we can consider a Pfaff form ω on the manifold M as a function on TM replacing ω by $\iota\omega$. In local coordinates we have

$$(4) \quad (\iota\omega)(x, y) = \sum \omega_i y^i, \text{ if } \omega = \sum \omega_i dx^i.$$

Definition. If $\tilde{D}$ is a distribution on the tangent bundle TM and ω is a Pfaff form on M , then we say that ω is $\tilde{D}$ -convex Pfaff form if the function $\iota\omega$ is a $\tilde{D}$ -convex function on TM .

Proposition 1. Every Pfaff form on a Riemannian manifold is vertical-convex.

Proof. Using the fact that the vertical distribution V is generated by the vertical lifts of the vector fields on M and by means the formula $[X^*, Y^*] = 0$, for any vector fields X, Y on M , it follows that the distribution V on TM is completely integrable. Thus we can apply Proposition 1.3 for the function $\iota\omega: TM \rightarrow R$ and the foliation V . To obtain the assertion of our proposition we remark that for every point $z = (x, y) \in TM$, the leaf F_z of the foliation V is $F_z = \pi^{-1}(\pi(z))$ and therefore $\iota\omega/F_z$ is a linear function $(\iota\omega)_x(y) = \sum \omega_i(x) y^i$. On the other hand the Levi-Civita connection $\tilde{\nabla}/F_z$,

induced on the leaf F_z , by the Levi-Civita connection $\tilde{\Delta}$ of the Riemannian manifold TM is a locally Euclidean connection. It follows that $\iota\omega/F_z$ is a convex function, for every $z \in TM$. As we stated above, using Proposition 1.3, it follows that $\iota\omega$ is a V -convex function, that is ω is a vertical-convex Pfaff form.

Proposition 2. A Pfaff form $\omega = \sum \omega_i dx^i$ on a Riemannian manifold M is horizontal-convex if and only if ω verifies the following local equations:

$$(5) \quad \frac{\partial^2 \omega_h}{\partial x^i \partial x^j} = \sum_k \left(\Gamma_{jh}^k \frac{\partial \omega_k}{\partial x^i} + \Gamma_{ih}^k \frac{\partial \omega_k}{\partial x^j} + \Gamma_{ij}^k \frac{\partial \omega_h}{\partial x^k} \right) + \sum_k \left(\frac{\partial \Gamma_{jh}^k}{\partial x^i} - \frac{\partial \Gamma_{ih}^k}{\partial x^j} - \sum_m \Gamma_{ih}^m \Gamma_{jm}^k - \sum_m \Gamma_{ij}^m \Gamma_{mh}^k \right) \omega_k, \quad h, i, j = 1, 2, \dots, n.$$

Proof. The horizontal distribution H on TM is generated by the horizontal lifts of the vector fields on M , such that we can write for every $\tilde{X} \in H$:

$$(6) \quad \tilde{X} = \sum_i X^i \left(\frac{\partial}{\partial x^i} \right)^h, \quad \text{with } X^i \in C^\infty(TM, R).$$

For the function $\omega : TM \rightarrow R$ we have

$$(7) \quad \text{Hess}(\omega)(\tilde{X}, \tilde{X}) = \sum_{i,j} X^i X^j \text{Hess}(\omega) \left(\left(\frac{\partial}{\partial x^i} \right)^h, \left(\frac{\partial}{\partial x^j} \right)^h \right)$$

and

$$(8) \quad \text{Hess}(\omega) \left(\left(\frac{\partial}{\partial x^i} \right)^h, \left(\frac{\partial}{\partial x^j} \right)^h \right) = \left(\frac{\partial}{\partial x^i} \right)^h \left(d(\omega) \left(\frac{\partial}{\partial x^j} \right)^h \right) - d(\omega) \left(\tilde{\nabla} \left(\frac{\partial}{\partial x^i} \right)^h, \left(\frac{\partial}{\partial x^j} \right)^h \right).$$

From [3]

$$(9) \quad \left(\frac{\partial}{\partial x^j} \right)^h = \frac{\partial}{\partial x^j} - \sum_{i,m} \Gamma_{ji}^m y^i \frac{\partial}{\partial y^m}$$

and because

$$(10) \quad d(\omega) = \sum_{h,k} \frac{\partial \omega_h}{\partial x^k} y^h dx^k + \sum_k \omega_k dy^k,$$

we obtain

$$(11) \quad \begin{aligned} \left(\frac{\partial}{\partial x^i} \right)^h \left(d(\omega) \left(\frac{\partial}{\partial x^j} \right)^h \right) &= \sum_h \left(\frac{\partial^2 \omega_h}{\partial x^i \partial x^j} - \sum_k \Gamma_{jh}^k \frac{\partial \omega_k}{\partial x^i} - \sum_k \Gamma_{ih}^k \frac{\partial \omega_k}{\partial x^j} - \right. \\ &\quad \left. - \sum_k \frac{\partial \Gamma_{jh}^k}{\partial x^i} \omega_k + \sum_{m,k} \Gamma_{ih}^m \Gamma_{jm}^k \omega_k \right) y^h. \end{aligned}$$

Now we compute

$$\tilde{\nabla} \left(\frac{\partial}{\partial x^i} \right)^h \left(\frac{\partial}{\partial x^j} \right)^h = \left(\nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right)^h - \frac{1}{2} vR \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right),$$

where $R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = R_{ijk}^h \frac{\partial}{\partial x^h} \otimes dx^k$ and therefore [3]

$$vR \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \sum_{h,k} R_{ijk}^h y^k \frac{\partial}{\partial y^h},$$

that is

$$(12) \quad \tilde{\nabla} \left(\frac{\partial}{\partial x^i} \right)^h \left(\frac{\partial}{\partial x^j} \right)^h = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} - \sum_{k,l,m} \Gamma_{ij}^k \Gamma_{kl}^m y^l \frac{\partial}{\partial y^m} - \frac{1}{2} \sum_{h,k} R_{ijk}^h y^k \frac{\partial}{\partial y^h}.$$

Since the tensor field R_{ijk}^h is antisymmetric with respect to the indexes i and j , we have

$$(13) \quad \sum_{i,j} R_{ijk}^h X^i X^j = 0, \quad \forall h, k = 1, \dots, n.$$

Using the formulas (7), (10), (11), (12) and (13), we obtain

$$(14) \quad \text{Hess}(\omega)(\tilde{X}, \tilde{X}) = \sum_{i,j,h} \left(\frac{\partial^2 \omega_h}{\partial x^i \partial x^j} - \sum_k \left(\Gamma_{jh}^k \frac{\partial \omega_k}{\partial x^i} + \Gamma_{ih}^k \frac{\partial \omega_k}{\partial x^j} + \Gamma_{ij}^k \frac{\partial \omega_h}{\partial x^k} \right) - \sum_k \left(\frac{\partial \Gamma_{jh}^k}{\partial x^i} - \sum_m \Gamma_{ih}^m \Gamma_{jm}^k \right) \omega_k \right) X^i X^j y^h.$$

If we consider a point $z = (x, y) \in TM$ and a horizontal vector $\tilde{X}_z = \sum_i X^i \left(\frac{\partial}{\partial x^i} \right)_z$ and then the point $z' = (x, -y)$ and the horizontal vector $\tilde{X}_{z'} = \sum_i X^i \left(\frac{\partial}{\partial x^i} \right)_{z'}$, where $X^i, i = 1, \dots, n$ are the same as for $\tilde{X}_z$, then by (14)

we have $\text{Hess}(\omega)(\tilde{X}_z, \tilde{X}_z) \geq 0$ and $\text{Hess}(\omega)(\tilde{X}_{z'}, \tilde{X}_{z'}) \geq 0$, if and only if ω verifies the local equations (5).

Remark. The horizontal-convex Pfaff forms are also horizontal-concave.

Corollary. For a locally Euclidean space M , a Pfaff form ω is horizontal-convex if and only if ω is „local affine“, that is

$$(15) \quad \omega = \sum_{i,j} a_{ij} x^i dx^j + \sum_i a_i dx^i, \quad \text{where } a_{ij} \text{ and } a_i \text{ are constant.}$$

3. The Case of the Complete Lift of a Smooth Function. If $f: M \rightarrow \mathbb{R}$ is a smooth function, then the function $\iota(df)$ is denoted by f° and it is called the complete lift of f to the tangent bundle.

For a Riemannian manifold M , by Proposition 1 we obtain

Proposition 3. For every smooth function $f: M \rightarrow \mathbb{R}$, his complete lift is a vertical-convex function.

Also by Proposition 2 we obtain

Proposition 4. If $f: M \rightarrow \mathbb{R}$ is a smooth function, then his complete lift is a horizontal-convex function if and only if f verifies the following local equations:

$$(16) \quad \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^h} = \sum \left(\Gamma_{jh}^k \frac{\partial^2 f}{\partial x^i \partial x^k} + \Gamma_{ih}^k \frac{\partial^2 f}{\partial x^j \partial x^k} + \Gamma_{ij}^k \frac{\partial^2 f}{\partial x^h \partial x^k} \right) + \\ + \sum_k \left(\frac{\partial \Gamma_{jh}^k}{\partial x^i} - \sum_m \Gamma_{ih}^m \Gamma_{jm}^k - \sum_m \Gamma_{ij}^m \Gamma_{mh}^k \right) \frac{\partial f}{\partial x^k}.$$

Corollary. *If M is a locally Euclidean space, then the complete lift of a smooth function $f: M \rightarrow R$ is a horizontal-convex function if and only if f is a „local affine quadratic form“, that is*

$$(17) \quad f(x) = \sum_{i,j} a_{ij} x^i x^j + \sum_i a_i x^i + a_0, \quad \text{where } a_{ij}, a_i, a_0 \text{ are constant.}$$

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UNELE FUNCȚII VERTICAL ȘI ORIZONTAL CONVEXE PE FIBRATUL TANGENT AL UNEI VARIETĂȚI RIEMANNIENE

(Rezumat)

Se studiază convexitatea verticală și orizontală a unei forme Pfaff pe o varietate Riemanniană M considerată ca funcție pe TM . În particular se cercetează liftul complet al unei funcții.

ON THE CURVATURE OF SEMI-INVARIANT SUBMANIFOLDS IN A SASAKIAN SPACE FORM

BY

AUREL BEJANCU and N. PAPAGHIUC

0. Introduction. Recently, we have introduced and studied in [2] the class of semi-invariant submanifolds in a Sasakian manifold. Roughly speaking, a semi-invariant submanifold in a Sasakian manifold is the notion corresponding to what is known as a CR-submanifold in a Kaehler manifold [1]. For the right definition see the next paragraph.

The purpose of the present paper is to study some problems on the curvatures of semi-invariant submanifolds in a Sasakian space form. In §2 we introduce the notion of F -sectional curvature of a semi-invariant submanifold and by means of it we study D -geodesic semi-invariant submanifolds. Since a proper semi-invariant submanifold of a Sasakian manifold is never totally geodesic [2], we study further $D^\perp$ -geodesic and $(D, D^\perp)$ -geodesic semi-invariant submanifolds. In Theorem 6 we give a complete characterization of semi-invariant submanifolds of constant sectional curvature in a Sasakian space form. In the last section of the paper we obtain properties of the Ricci tensor and scalar curvature of a semi-invariant submanifold.

1. Semi-invariant Submanifolds of a Sasakian Manifold. Let $\tilde{M}$ be a $2n+1$ -dimensional almost contact metric manifold. Denote by (F, ξ, η, g) the structure tensors of $\tilde{M}$, where F is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form respectively and g is the Riemannian metric on $\tilde{M}$. These tensor fields are related by

$$(1.1) \quad F^2X = -X + \eta(X)\xi; \quad F\xi = 0; \quad \eta(FX) = 0; \quad \eta(\xi) = 1$$

and

$$(1.2) \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y tangent to $\tilde{M}$.

Throughout the paper we suppose $\tilde{M}$ is a Sasakian manifold, that is, [3, p. 73]

$$(1.3) \quad (\tilde{\nabla}_X F)Y = g(X, Y)\xi - \eta(Y)X, \quad (1.4)$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.

Let M be a m -dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. Suppose the structure vector field ξ is tangent to M . Denote by TM and $TM^\perp$ respectively the tangent bundle of M and the normal bundle to M . The 1-dimensional distribution spanned by ξ is denoted by $\{\xi\}$. Suppose there exists on M two differentiable orthogonal distributions D and $D^\perp$ such that the following conditions are fulfilled:

- (i) $TM = D \oplus D^\perp \oplus \{\xi\}$;
- ii) the distribution D is invariant by F , that is, $F(D_x) = D_x$ for each $x \in M$;
- (iii) the distribution $D^\perp$ is anti-invariant by F , that is, $F(D_x^\perp) \subset T_x M^\perp$ for each $x \in M$.

Definition The submanifold M of the Sasakian manifold $\tilde{M}$ endowed with the pair of distributions $(D, D^\perp)$ satisfying conditions (i), (ii) and (iii) is called a semi-invariant submanifold of $\tilde{M}$.

The distributions D and $D^\perp$ are called respectively the invariant distribution and the anti-invariant distribution on M . Suppose the dimension of D_x (respectively $D_x^\perp$) be $2p$ (respectively q). Then it is easily seen that when $p=0$ (respectively $q=0$) the semi-invariant submanifold becomes an anti-invariant submanifold (respectively an invariant submanifold).

We denote by the same symbol g both metrics on M and $\tilde{M}$. The projection morphisms of TM to D and $D^\perp$ are denoted respectively by P and Q . Then we have

$$(1.4) \quad X = PX + QX + \eta(X)\xi$$

for any vector field X tangent to M .

Denote by $\Gamma(D)$ the module of all differentiable sections of the vector bundle D on M . We make use of such notation for any other vector bundle on M .

Let ∇ be the Levi-Civita connection on M and h be the second fundamental form of M . Then, the equations of Gauss and Weingarten are given by

$$(1.5) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

and respectively

$$(1.6) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$, where $\nabla^\perp$ is the normal connection induced by $\tilde{\nabla}$ on $TM^\perp$ and A_N is the fundamental tensor of Weingarten with respect to the normal section N .

Denote by $\tilde{R}$, R and $R^\perp$ the curvature tensor field of $\tilde{M}$, M and respectively the curvature tensor field of the normal connection $\nabla^\perp$. Then the structure equations of Gauss and Ricci are given by

$$(1.7) \quad g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W)),$$

$$(1.8) \quad g(\tilde{R}(X, Y)N, \bar{N}) = g(R^\perp(X, Y)N, \bar{N}) - g([A_N, A_{\bar{N}}](X), Y),$$

for all $X, Y, Z, W \in \Gamma(TM)$ and $N, \bar{N} \in \Gamma(TM^\perp)$.

For the next section we need

Lemma 1 [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then we have*

$$(1.9) \quad h(X, \xi) = 0 \quad \text{for any } X \in \Gamma(D) \text{ and}$$

$$(1.10) \quad h(Y, \xi) = -FY \quad \text{for any } Y \in \Gamma(D^\perp).$$

2. Sectional Curvature of a Semi-invariant Submanifold in a Sasakian Space Form. Let $\tilde{M}(c)$ be a $2n+1$ -dimensional Sasakian space form of constant F -sectional curvature c . Then, the curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by [3, p. 97]

$$(2.1) \quad \begin{aligned} \tilde{R}(X, Y)Z = & \frac{c+3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{c-1}{4} \{\eta(X)\eta(Z)Y - \\ & - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(Z, FY)FX - \\ & - g(Z, FX)FY + 2g(X, FY)FZ\} \end{aligned}$$

for all X, Y, Z tangent to $\tilde{M}(c)$.

Denote by $K_M(X \wedge Y)$ the sectional curvature of M determined by orthonormal vectors X and Y tangent to M , where M is a semiinvariant submanifold of $\tilde{M}(c)$. Then by using (1.7) and (2.1) we obtain

$$(2.2) \quad K_M \in (X \wedge Y) = \frac{c+3}{4} + \frac{c-1}{4} \{3g(PX, FPY)^2 - \eta(X)^2 - \eta(Y)^2\} + \\ + g(h(X, X), h(Y, Y)) - g(h(X, Y), h(X, Y)).$$

Definition. The F -sectional curvature $H(X)$ of the semi-invariant submanifold M determined by the unit vector $X \in \Gamma(D)$ is the sectional curvature of M determined by $\{X, FX\}$.

From (2.2) follows

$$(2.3) \quad H(X) = c + g(h(X, X), h(FX, FX)) - g(h(X, FX), h(X, FX)).$$

We recall

Lemma 2 [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if we have*

$$(2.4) \quad h(X, FY) = h(FX, Y) \quad \text{for all } X, Y \in \Gamma(D).$$

By using this Lemma we have

Proposition 1. *Let M be a semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. If the distribution $D \oplus \{\xi\}$ is involutive then $H(X) \leq c$ for all unit vector fields $X \in \Gamma(D)$.*

Proof. From (2.4) follows

$$(2.5) \quad h(FX, FY) + h(X, Y) = 0$$

for all $X, Y \in \Gamma(D)$. Then by using (2.5) and (2.3) we obtain the assertion of the proposition.

Definitions. 1. The semi-invariant submanifold M is called D -geodesic (respectively $D^\perp$ -geodesic) if we have $h(X, Y) = 0$ for all $X, Y \in \Gamma(D)$ (respectively $h(X, Y) = 0$ for all $X, Y \in \Gamma(D^\perp)$).

2. The semi-invariant submanifold M is called $(D, D^\perp)$ -geodesic if $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$.

Now, we state

Theorem 1. A semi-invariant submanifold M of $\bar{M}(c)$ is D -geodesic if and only if the following conditions are fulfilled:

(i) $\tilde{D} \oplus \{\xi\}$ is an involutive distribution;

(ii) $H(X) = c$ for any $X \in \Gamma(D)$.

Proof. Suppose M is a D -geodesic semi-invariant submanifold. Then from (2.3) follows $H(X) = c$ and the involutivity condition is obviously satisfied.

Conversely, supposing conditions (i) and (ii) be satisfied from (2.3) and (2.5) follows $h(X, X) = 0$ for any $X \in \Gamma(D)$. This is sufficient for vanishing of h on the invariant distribution D .

We choose the local field of orthonormal frames

$$(2.6) \quad \{E_1, \dots, E_p, E_{p+1} = FE_1, \dots, E_{2p} = FE_p, E_{2p+1}, \dots, E_{2p+q}, \xi\}$$

on M such that $E_i \in \Gamma(D)$ ($i = 1, 2, \dots, p$) and $E_{2p+a} \in \Gamma(D^\perp)$ ($a = 1, 2, \dots, q$).

Definition. The semi-invariant submanifold M of $\bar{M}$ is called D -minimal (respectively $D^\perp$ -minimal) if we have

$$\sum_{i=1}^{2p} \{h(E_i, E_i)\} = 0 \quad (\text{respectively } \sum_{a=1}^q \{h(E_{2p+a}, E_{2p+a})\} = 0).$$

Theorem 2. Let M be a $D^\perp$ -minimal semi-invariant submanifold of $\bar{M}(c)$ with $q > 1$. Then M is $D^\perp$ -geodesic if and only if we have

$$(2.7) \quad K_M(X \wedge Y) = \frac{c+3}{4}$$

for all $X, Y \in \Gamma(D^\perp)$.

Proof. Suppose M is $D^\perp$ -geodesic. Then from (2.2) follows (2.7) since $D^\perp$ belongs to the contact distribution of $\bar{M}(c)$. Conversely, from (2.2) taking account of (2.7) we obtain

$$(2.8) \quad g(h(X, X), h(Y, Y)) - g(h(X, Y), h(X, Y)) = 0$$

for all $X, Y \in \Gamma(D^\perp)$. Then, by means of the $D^\perp$ -minimality of M from (2.8) follows $h(E_{2p+a}, E_{2p+b}) = 0$ for all $a, b = 1, 2, \dots, q$. Thus M is $D^\perp$ -geodesic.

We denote by $\|h_{D^\perp}\|$ the differentiable function on M defined by

$$\|h_{D^\perp}\|^2 = \sum_{a,b=1}^q \{g(h(E_{2p+a}, E_{2p+b}), h(E_{2p+a}, E_{2p+b}))\}.$$

As we have seen in Theorem 2 for a class of $D^\perp$ -minimal semi-invariant submanifolds we have $\|h_{D^\perp}\| = 0$. The next theorem establishes another class of semi-invariant submanifolds on which $\|h_{D^\perp}\|$ is a non-null constant.

Theorem 3. Let M be a D^1 -minimal semi-invariant submanifold of $\tilde{M}(c)$ with $q > 1$. If the sectional curvature of M satisfies

$$(2.9) \quad K_M(X \wedge Y) = \frac{c-1}{4} \quad \text{for all } X, Y \in \Gamma(D^1) \quad \text{then we have}$$

$$(2.10) \quad \|h_{D^1}\|^2 = q(q-1).$$

Proof. As a consequence of (2.2) and (2.9) we obtain

$$(2.11) \quad g(h(X, Y), h(X, Y)) - g(h(X, X), h(Y, Y)) = 1$$

for all $X, Y \in \Gamma(D^1)$. Then by a straightforward computation using (2.11) and the D^1 -minimality condition we obtain (2.10).

Theorem 4. Let M be a semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Suppose the sectional curvature of M satisfies

$$(2.12) \quad K_M(X \wedge Y) = \frac{c+3}{4}$$

or any $X \in \Gamma(D)$, $Y \in \Gamma(D^1)$ and one of the following conditions is fulfilled

- (i) M is D -minimal
- (ii) M is D^1 -minimal.

Then M is a (D, D^1) -geodesic semi-invariant submanifold.

The proof follows from (2.2) where we take $X \in \Gamma(D)$ and $Y \in \Gamma(D^1)$. It remains to study sectional curvatures of plane sections which contain the structure vector field ξ . From (2.2), by using (1.9) and (1.10) we obtain

Theorem 5. Let M be a semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then we have

$$(2.13) \quad K_M(X \wedge \xi) = 1 \quad \text{and}$$

$$(2.14) \quad K_M(Y \wedge \xi) = 0$$

for all unit vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^1)$.

From Theorem 5 we have

Theorem 6. Let M be a semi-invariant submanifold of constant sectional curvature immersed in a Sasakian space form $\tilde{M}(c)$. Then we have either

- (i) M is an elliptic space immersed as an invariant submanifold in $\tilde{M}(c)$ or
- (ii) M is a locally Euclidean space immersed as an anti-invariant submanifold in $\tilde{M}(c)$.

Corollary 1. There exists no proper semi-invariant submanifolds of constant sectional curvature immersed in a Sasakian space form.

Definitions. 1. The second fundamental form of M is called commutative if we have $[A_N, A_{\bar{N}}] = 0$ for all $N, \bar{N} \in \Gamma(TM^1)$.

2. The anti-invariant distribution D^1 is said to be flat if we have $\nabla_X Y \in \Gamma(D^1)$ for any $X, Y \in \Gamma(D^1)$.

Now, we give a general result on semi-invariant submanifolds of a Sasakian manifold.

Proposition 2. *Let M be a generic semi-invariant submanifold of a Sasakian manifold $\tilde{M}$ with flat anti-invariant distribution. Then, M is a $(D, D^\perp)$ -geodesic semi-invariant submanifold.*

Proof. We take $X, Y \in \Gamma(D^\perp)$ and from equations of Gauss and Weingarten obtain

$$(2.15) \quad Fh(X, Y) + A_{FY}X + g(X, Y)\xi = 0$$

and

$$(2.16) \quad \nabla_X^{\perp} FY - F\nabla_X Y = 0.$$

Now, for each $N \in \Gamma(TM^\perp)$ there exists $Y \in \Gamma(D^\perp)$ such that $N = FY$. Then, by using (2.15) we have

$$g(h(X, Z), N) = g(A_{FY}X, Z) = -g(Fh(X, Y), Z) = 0 \text{ for all } X \in \Gamma(D^\perp) \text{ and } Z \in \Gamma(D).$$

Hence M is $(D, D^\perp)$ -geodesic.

Finally, we state

Theorem 7. *Let M be a generic semi-invariant submanifold immersed in $\tilde{M}(c)$ of codimension $q > 1$. Suppose the second fundamental form of M is commutative and the anti-invariant distribution is flat. Then we have*

$$(2.17) \quad K_M(X \wedge Y) = \frac{c-1}{4} \text{ for all } X, Y \in \Gamma(D^\perp).$$

Proof. Taking account of (2.1) and Ricci equation (1.8) we obtain

$$(2.18) \quad g(R^\perp(X, Y)N, \bar{N}) = \frac{c-1}{4} \{g(N, FQY)g(\bar{N}, FQX) - g(N, FQX)g(\bar{N}, FQY)\}$$

for all $X, Y \in \Gamma(TM)$ and $N, \bar{N} \in \Gamma(TM^\perp)$. From (2.16) follows

$$(2.19) \quad FR(X, Y)Z = R^\perp(X, Y)FZ \text{ for all } X, Y, Z \in \Gamma(D^\perp).$$

Since M is a generic semi-invariant submanifold, there exists $Z, W \in \Gamma(D^\perp)$ such that $N = FZ$ and $\bar{N} = FW$. Then by (2.19) we see that (2.18) becomes

$$(2.20) \quad g(R(X, Y)Z, W) = \frac{c-1}{4} \{g(Z, Y)g(W, X) - g(Z, X)g(W, Y)\}$$

for all $X, Y, Z, W \in \Gamma(D^\perp)$. Thus (2.17) follows from (2.20).

3. Ricci Tensor and Scalar Curvature of a Semi-invariant Submanifold.

Let M be a semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. If $\{E_1, \dots, E_m\}$ is a local field of orthonormal frames then it is known that Ricci tensor S of M is given by

$$(3.1) \quad S(X, Y) = \sum_{i=1}^m \{g(R(E_i, X)Y, E_i)\} \text{ for } X, Y \in \Gamma(TM).$$

Then, by a direct computation using Gauss equation (1.7) and the distinguished field of frames (2.6) we obtain

$$(3.2) \quad S(X, Y) = \frac{(m-1)(c+3) - (c-1)}{4} g(X, Y) - \frac{(c-1)(m-2)}{4} \eta(X)\eta(Y) + \\ + \frac{3(c-1)}{4} g(PX, PY) + \sum_{i=1}^m \{g(h(X, Y), h(E_i, E_i)) - g(h(X, E_i), h(Y, E_i))\}.$$

If $\{N_\alpha\}$ ($\alpha=1, 2, \dots, 2n+1-m$) is a local field of orthonormal sections of $TM^\perp$, then we have

$$(3.3) \quad h(X, Y) = \sum_{\alpha=1}^{2n+1-m} g(A_\alpha X, Y) N_\alpha,$$

where $A_\alpha = A_{N_\alpha}$. Thus, by (3.3) we see that (3.2) becomes

$$(3.4) \quad S(X, Y) = \frac{(m-1)(c+3) - (c-1)}{4} g(X, Y) + \frac{3(c-1)}{4} g(PX, PY) - \\ - \frac{(c-1)(m-2)}{4} \eta(X)\eta(Y) + \sum_{\alpha=1}^{2n+1-m} \{(\text{trace } A_\alpha) g(A_\alpha X, Y) - g(A_\alpha X, A_\alpha Y)\}.$$

Then the scalar curvature ρ of M is given by

$$(3.5) \quad \rho = \frac{m(m-1)(c+3) - 2(m-3p-1)(c-1)}{4} + m^2 \|H\|^2 - \|h\|^2$$

where $H = \frac{1}{m} \sum_{\alpha=1}^{2n+1-m} (\text{trace } A_\alpha) N_\alpha$ is the mean curvature vector of M and $\|h\|$ is the length of the second fundamental form of M .

From (3.4) and (3.5) we obtain

Theorem 8. *Let M be a minimal semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then we have*

$$(3.6) \quad S - \frac{(c+3)(m-1) - (c-1)}{4} g - \frac{3(c-1)}{4} g^0(P \times P) + \frac{(c-1)(m-2)}{4} \eta \otimes \eta$$

is negative semi-definite and

$$(3.7) \quad \rho \leq \frac{m(m-1)(c+3) - 2(m-3p-1)(c-1)}{4}.$$

Remark. The equality in (3.7) is possible only when M is an invariant submanifold of $\tilde{M}(c)$.

It is well known [3, p. 99] that R^{2n+1} is endowed with a Sasakian structure of constant F -sectional curvature $c = -3$. Then, by using (3.7) we have

Theorem 9. *Let M be a minimal semi-invariant submanifold of R^{2n+1} such that $p = q$. Then, the scalar curvature of M is non-positive.*

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ASUPRA CURBURII SUBVARIETĂȚILOR SEMI-INVARIANTE ALE UNEI FORME SPAȚIALE SASAKIENE

(Rezumat)

Se continuă lucrarea [2] obținând rezultate referitoare la curbura secțională, curbura scalară și forma a doua fundamentală a unei subvarietăți semi-invariante într-o formă spațială sasakiană.

ON SOME REMARKABLE GROUPS OF QUASI-CONNECTION TRANSFORMATIONS

BY

ELENA VAMANU

1. Introduction. Let M be a C^∞ -differentiable, n -dimensional manifold, $n > 1$, with an almost product structure defined by the tensor field F_j^i . In [3] there are determined all the quasi-connections in which F_j^i is covariantly constant. These quasi-connections were called F -quasi-connections. They are given by

$$(1) \quad \bar{\Gamma}_{kh}^i = \Gamma_{kh}^i + W_{hk}^i,$$

where Γ_{kh}^i is the F -quasi-connection

$$(2) \quad \Gamma_{kh}^i = F_{kh}^m (\gamma_{mh}^i + \frac{1}{2} F_j^i \nabla_m F_h^j),$$

γ_{mh}^i is an arbitrary but fixed connection on M , ∇ is the operator of covariant differential with respect to γ_{mh}^i , and W_{hk}^i is an arbitrary tensor field that satisfies the condition

$$(3) \quad Q_{mh}^{ki} W_{kj}^h = W_{mj}^i;$$

here Q_{mh}^{ki} denotes the Obata operator defined by

$$(4) \quad Q_{mh}^{ki} = \frac{1}{2} (\delta_m^k \delta_h^i + F_m^k F_h^i)$$

[see [3], [4] for details].

Now, if we consider the transformations $\Gamma_{kh}^i \rightarrow \bar{\Gamma}_{kh}^i$ defined by (1), W_{hk}^i being an arbitrary tensor field that satisfies (3), the set of these transformations is a group G , isomorphic to the additive group of the tensor field W .

For certain choices of W we obtain certain subgroups of the group G . In [4] there are examined the subgroups G_1 and G_2 given by

$$(5) \quad W_{kj}^i = \delta_k^i p_j$$

and

$$(6) \quad W_{kj}^i = F_k^i p_j$$

respectively.

In the present paper we aim to determine a family $G(\alpha, \beta)$ of subgroups of the group G and to find some of its invariants.

2. Subgroups $G(\alpha, \beta)$. Let us consider

$$(7) \quad W_{kj}^i = (\alpha \delta_k^i + \beta F_k^i) p_j,$$

where p_j is an arbitrary covector field and α, β , are two arbitrary constants meeting condition

$$\alpha^2 - \beta^2 \neq 0.$$

One easily verifies that W defined by (7) satisfies (3) and hence we obtain the transformations $\Gamma \rightarrow \bar{\Gamma}$ given by

$$(8) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + (\alpha \delta_k^i + \beta F_k^i) p_j.$$

The set of these transformations defines the subgroup $G(\alpha, \beta)$ of the group G . If we let vary α and β we obtain a family of subgroups of G .

Remark 1. For $\alpha=1, \beta=0$, and $\alpha=0, \beta=1$, we get the subgroups G_1 and G_2 respectively, studied in [4].

Let $\Gamma_{jk}^i, \bar{\Gamma}_{jk}^i$ be two F -quasi-connections of type (8) with $T_{jk}^i, \bar{T}_{jk}^i$ be their torsion tensors, and $R_{jkh}^i, \bar{R}_{jkh}^i$ their curvature tensors.

Using (8) and the expressions of these tensors given in [2], we obtain

$$(9) \quad \bar{T}_{jk}^i = T_{jk}^i + (\alpha \delta_k^i + \beta F_k^i) p_j - (\alpha \delta_j^i + \beta F_j^i) p_k,$$

$$(10) \quad \bar{R}_{hij}^k = R_{hij}^k + (\alpha \delta_h^k + \beta F_h^k) (D_i p_j - D_j p_i + T_{ij}^m p_m),$$

where D is the operator of covariant differential with respect to Γ_{jk}^i .

By contracting i, k in (9) we get

$$(11) \quad \bar{T}_{ji}^i = T_{ji}^i + \alpha(n-1)p_j + \beta(F_i^i p_j - F_j^i p_i).$$

If we put

$$(12) \quad \bar{T}_j = \bar{T}_{ji}^i, \quad T_j = T_{ji}^i, \quad f = F_i^i, \quad q_j = F_j^i p_i,$$

(11) becomes

$$(13) \quad \bar{T}_j = T_j + \alpha(n-1)p_j + \beta(fp_j - q_j).$$

Now, multiplying (9) by F_i^m , we get

$$(14) \quad \bar{T}_{jk}^i F_i^m = T_{jk}^i F_i^m + (\alpha F_k^m + \beta \delta_k^m) p_j - (\alpha F_j^m + \beta \delta_j^m) p_k.$$

If we contract m, k in (14) we obtain

$$(15) \quad \bar{T}_{jk}^i F_i^k = T_{jk}^i F_i^k + \beta(n-1)p_j + \alpha(fp_j - q_j).$$

From (13) and (15) we have

$$(16) \quad p_j = \frac{\alpha(\bar{T}_j - T_j) - \beta(\bar{T}_{jk}^i - T_{jk}^i)F_i^k}{(\alpha^2 - \beta^2)(n-1)}.$$

Substituting (16) in (9) we get a first type of invariant tensors of the groups $G(\alpha, \beta)$, namely

$$(17) \quad {}_1I_{jk}^i(\alpha, \beta) = T_{jk}^i + \frac{(\alpha \delta_k^i + \beta F_k^i)(-\alpha T_j + \beta T_{jm}^h F_h^m) - (\alpha \delta_j^i + \beta F_j^i)(-\alpha T_k + \beta T_{km}^h F_h^m)}{(\alpha^2 - \beta^2)(n-1)}.$$

Remark 2. For $\alpha=1, \beta=0$ and $\alpha=0, \beta=1$ we have, respectively

$$(18) \quad {}_1I_{jk}^i(1,0) = T_{jk}^i + \frac{1}{n-1} (\delta_j T_k - \delta_k T_j),$$

$$(19) \quad {}_1I_{jk}^i(0,1) = T_{jk}^i + \frac{1}{n-1} (F_j T_{km}^h - F_k T_{jm}^h) F_h^m,$$

that is, the invariant tensor of this type for the groups G_1, G_2 obtained in [4].

By contracting k, h in (10) we get

$$(20) \quad \bar{R}_{kij}^k = R_{kij}^k + (\alpha n + \beta f) (D_i p_j - D_j p_i + T_{ij}^m p_m).$$

Denoting

$$(21) \quad \bar{P}_{ij} = \bar{R}_{kij}^k, \quad P_{ij} = R_{kij}^k,$$

(20) becomes

$$(22) \quad \bar{P}_{ij} = P_{ij} + (\alpha n + \beta f) (D_i p_j + T_{ij}^m p_m).$$

Hence, from (10) and (22), we find a second type of invariant tensors of the groups $G(\alpha, \beta)$, given by

$$(23) \quad {}_2I_{hij}^k(\alpha, \beta) = R_{hij}^k - \frac{1}{\alpha n + \beta f} (\alpha \delta_h^k + \beta F_h^k) P_{ij}.$$

Remark 3. For $\alpha=1, \beta=0$ and $\alpha=0, \beta=1$, we have, respectively

$$(24) \quad {}_2I_{hij}^k(1,0) = R_{hij}^k - \frac{1}{n} \delta_h^k P_{ij},$$

$$(25) \quad {}_2I_{hij}^k(0,1) = R_{hij}^k - \frac{1}{f} F_h^k P_{ij}.$$

Multiplying (10) by F^h we get

$$(26) \quad \bar{R}_{hij}^k F_t^h = R_{hij}^k F_t^h + (\alpha F_t^k + \beta \delta_t^k) (D_i p_j - D_j p_i + T_{ij}^m p_m).$$

If we contract k, t in (26) we obtain

$$(27) \quad \bar{R}_{hij}^k F_k^h = R_{hij}^k F_k^h + (\alpha f + \beta n) (D_i p_j - D_j p_i + T_{ij}^m p_m),$$

from which it follows

$$(28) \quad D_i p_j - D_j p_i + T_{ij}^m p_m = \frac{\bar{R}_{hij}^k F_k^h - R_{hij}^k F_k^h}{\alpha f + \beta n}.$$

Substituting (28) in (10) we obtain a third type of invariant tensors of the groups $G(\alpha, \beta)$:

$$(29) \quad {}_2I_{hij}^k(\alpha, \beta) = R_{hij}^k - \frac{1}{\alpha f + \beta n} (\alpha \delta_h^k + \beta F_h^k) R_{lij}^m F_m^l.$$

Remark 4. For $\alpha=1, \beta=0$, and $\alpha=0, \beta=1$, we have, respectively

$$(30) \quad {}_2I_{hij}^k(1, 0) = R_{hij}^k - \frac{1}{f} \delta_h^k R_{lij}^m F_m^l,$$

$$(31) \quad {}_2I_{hij}^k(0, 1) = R_{hij}^k - \frac{1}{n} F_h^k R_{lij}^m F_m^l.$$

Now, contracting k, j in (10) and denoting

$$(32) \quad \bar{R}_{hi} = \bar{R}_{hik}, \quad R_{hi} = R_{hik},$$

we get

$$(33) \quad \bar{R}_{hi} = R_{hi} + (\alpha \delta_h^k + \beta F_h^k) (D_i p_k - D_k p_i + T_{ik}^m p_m).$$

Multiplying (33) by F_j^h we have

$$(34) \quad \bar{R}_{hi} F_j^h = R_{hi} F_j^h + (\alpha F_j^k + \beta \delta_j^k) (D_i p_k - D_k p_i + T_{ik}^m p_m).$$

From (33) and (34) it follows

$$(35) \quad D_i p_h - D_h p_i + T_{ih}^m p_m = \frac{\alpha(\bar{R}_{hi} - R_{hi}) - \beta(\bar{R}_{mi} - R_{mi}) F_h^m}{\alpha^2 - \beta^2}.$$

Using (10) and (35) we obtain a fourth type of invariant tensors of the groups $G(\alpha, \beta)$ given by

$$(36) \quad {}_4I_{hij}^k(\alpha, \beta) = R_{hij}^k - \frac{1}{\alpha^2 - \beta^2} (\alpha \delta_h^k + \beta F_h^k) (\alpha R_{ji} - \beta R_{mi} F_j^m).$$

Remark 5.

$$(37) \quad {}_4I_{hij}^k(1, 0) = R_{hij}^k - \delta_h^k R_{ji},$$

$$(38) \quad {}_4I_{hij}^k(0, 1) = R_{hij}^k - F_h^k F_j^m R_{mi}.$$

Finally we can find some new invariants of the groups $G(\alpha, \beta)$ but they are not tensors.

By contracting i, k in (8) and denoting

$$(39) \quad \bar{\Gamma}_j = \bar{\Gamma}_{ji}^i, \quad \Gamma_j = \Gamma_{ji}^i,$$

we find

$$(40) \quad p_j = \frac{\bar{\Gamma}_j - \Gamma_j}{\alpha n + \beta f}.$$

From (8) and (40) we get the invariants

$$(41) \quad {}_1\Omega_{jk}^i(\alpha, \beta) = \Gamma_{jk}^i - \frac{1}{\alpha n + \beta f} (\alpha \delta_k^i + \beta F_k^i) \Gamma_j.$$

Remark 6.

$$(42) \quad {}_1\Omega_{jk}^i(1, 0) = \Gamma_{jk}^i - \frac{1}{n} \delta_k^i \Gamma_j,$$

$$(43) \quad {}_1\Omega_{jk}^i(0, 1) = \Gamma_{jk}^i - \frac{1}{f} F_k^i \Gamma_j.$$

Now, multiplying (8) by F_m^k and contracting i, k we get

$$(44) \quad \rho_j = \frac{(\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i) F_i^k}{\alpha f + \beta n}.$$

From (8) and (44) there result the invariants

$$(45) \quad {}_2\Omega_{jk}^i(\alpha, \beta) = \Gamma_{jk}^i - \frac{1}{\alpha f + \beta n} (\alpha \delta_k^i + \beta F_k^i) \Gamma_{jm}^h F_h^m.$$

Remark 7. For $\alpha=1, \beta=0$, and $\alpha=0, \beta=1$, we have respectively

$$(46) \quad {}_2\Omega_{jk}^i(1, 0) = \Gamma_{jk}^i - \frac{1}{f} \delta_k^i \Gamma_{jm}^h F_h^m,$$

$$(47) \quad {}_2\Omega_{jk}^i(0, 1) = \Gamma_{jk}^i - \frac{1}{n} F_k^i \Gamma_{jm}^h F_h^m.$$

Remark 8. The conditions $\alpha f + \beta n \neq 0$ and $\alpha n + \beta f \neq 0$ follow from the assumption made in the beginning of Sect. 1 and 2.

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ASUPRA UNOR GRUPURI DE TRANSFORMĂRI DE CVASI-CONEXIUNE

(Rezumat)

Pornind de la grupul transformărilor de F -cvasi-conexiune (v. [3]), se determină o familie de subgrupuri ale acestui grup, depinzând de doi parametri precum și o serie de invarianți ai acestor subgrupuri.

Pentru anumite valori particulare ale celor doi parametri se obțin rezultatele din [4].

ON A CLASS OF FINSLER CONNECTIONS

BY

I. GHINEA

0. In [1], [2] we find the set of all Finsler connections compatible with almost complex structure.

In the present paper we are obtaining a new form of the formulae expressing these connections and then give the class of Finsler connections with the property that Kawaguchi almost complex connection coincides with the canonical almost complex connection of $F_j^i(x, y)$.

1. Let M be a $2n$ -dimensional differential Finsler manifold. An almost complex Finsler structure of M is defined by a field of Finsler tensors $F_j^i(x, y)$ (y being the support element in point x), with the property

$$(1.1) \quad F_j^i F_k^j = -\delta_k^i.$$

Let associate to the almost complex structure F the operators

$$(1.2) \quad O_{hi}^j = \frac{1}{2}(\delta_h^j \delta_i^j - F_h^j F_i^j), \quad O_{hi}^j = \frac{1}{2}(\delta_h^j \delta_i^j + F_h^j F_i^j),$$

with their known properties

$$(1.3) \quad O + O^* = \delta \delta, \quad O O = O, \quad O O^* = O^* O = O, \quad O^* O^* = O^*.$$

We consider now on M a Finsler connection $F\Gamma = (N_j^i, F_{jk}^i, C_{jk}^i)$, defined as in [2], together with its covariant differentiation algorithm in respect to that connection, for the tensor fields from M . The connection $F\Gamma$ will be denoted more simpler $F\Gamma = (N, F, C)$.

We are looking for the Finsler connections $F\Gamma = (N, F, C)$ having the property

$$(1.4) \quad F_{j|k}^i = 0, \quad F_j^i|_k = 0,$$

where $|$ and $/$ means the covariant h -derivative and covariant v -derivative with respect to the $F\Gamma$ connection.

Definition 1.1. A Finsler connection $F\Gamma = (N_j^i, F_{jk}^i, C_{jk}^i)$ is called compatible with the almost complex structure if conditions (1.4) are fulfilled. We are saying that such a $F\Gamma$ connection is an almost complex Finsler connection.

By means of covariant differentiation of (1.4) it follows immediately that the operators (1.2) are covariant constant.

After these explanations we are ready to determine all almost complex Finsler connections. With this purpose we extend to the almost complex Finsler connections, the method utilized by prof. R. Miron [4] for metrical Finsler connections.

Let be $F\bar{\Gamma} = (\bar{N}_j^i, \bar{F}_{jk}^i, \bar{C}_{jk}^i)$ an arbitrary but fixed Finsler connection, defined on M . Then any connection $F\Gamma = (N_j^i, F_{jk}^i, C_{jk}^i)$ on M can be expressed like

$$(1.5) \quad \begin{cases} N_j^i = \bar{N}_j^i - A_j^i, \\ F_{jk}^i = \bar{F}_{jk}^i + \bar{C}_{js}^i A_k^s - B_{jk}^i, \\ C_{jk}^i = \bar{C}_{jk}^i + \bar{C}_{js}^i A_k^s - D_{jk}^i, \end{cases}$$

where A_j^i , B_{jk}^i and D_{jk}^i are components of the tensor fields obtained as difference of the couple $(F\bar{\Gamma}, F\Gamma)$ [3].

The tensor fields A_j^i , B_{jk}^i , D_{jk}^i will be determined so that the connection $(N_j^i, F_{jk}^i, C_{jk}^i)$ given by (1.5) verifies (1.4); it follows

$$(1.6) \quad \begin{cases} -B_{hk}^i F_j^h + B_{jk}^h F_h^i = -F_{j|k}^i - A_k^r F_{j|_r}^i, \\ -D_{hk}^i F_j^h + D_{jk}^h F_h^i = -F_{j|_k}^i, \end{cases}$$

where $|$ and $|_r$ represents the covariant h -derivative and covariant v -derivative with respect to the fixed connection $F\bar{\Gamma}$.

Let multiply (1.6) by $\frac{1}{2} F_j^i$ and put in evidence the operator O_{ji}^{ih} ; we get the tensorial equations

$$(1.7) \quad \begin{cases} O_{ki}^{hi} B_{jk}^i = -\frac{1}{2} F_j^i (F_{j|_k}^i + A_k^r F_{j|_r}^i), \\ O_{hi}^{ij} D_{jk}^i = -\frac{1}{2} F_j^i F_{j|_k}^i, \end{cases}$$

which obviously satisfy the compatibility conditions.

The general solutions of (1.7) are

$$(1.8) \quad \begin{cases} B_{jk}^h = \frac{1}{2} F_j^p (F_{p|_k}^h + A_k^q F_{p|_q}^h) - O_{qj}^{hp} U_{pk}^q, \\ D_{jk}^h = -\frac{1}{2} F_j^p F_{p|_k}^h - O_{qj}^{hp} V_{pk}^q, \end{cases}$$

where A_j^i , U_{pk}^q and V_{pk}^q are arbitrary Finsler tensor fields.

Taking into account (1.8), Eqs. (1.5) result in

$$(1.9) \quad \begin{cases} N_j^i = \bar{N}_j^i - A_j^i, \\ F_{jk}^i = \bar{F}_{jk}^i + \bar{C}_{js}^i A_k^s + \frac{1}{2} F_j^p (F_{p|_k}^h + A_k^q F_{p|_q}^h) + O_{jq}^{ip} U_{pk}^q, \\ C_{jk}^i = \bar{C}_{jk}^i + \frac{1}{2} F_j^p F_{p|_k}^h + O_{jq}^{ip} V_{pk}^q. \end{cases}$$

We get the

Theorem. The set of all almost complex Finsler connections $F\Gamma = (N_j^i, F_{jk}^i, C_{jk}^i)$ is given by Eqs. (1.9), where:

- i) $F\bar{\Gamma} = (\bar{N}_j^i, \bar{F}_{jk}^i, \bar{C}_{jk}^i)$ is a fixed Finsler connection;
- ii) $|$ and $|_r$ means the covariant h -derivative and covariant v -derivative with respect to the fixed connection $F\bar{\Gamma}$;
- iii) A_j^i , U_{pk}^q , V_{pk}^q are arbitrary Finsler tensor fields.

Remark. By comparison with the formulae given in [1], Eqs. (1.9) present the peculiarity of not containing the usual derivative with respect to the support element y , but only the covariant derivative with respect to the fixed connection.

2. Among the set of almost complex Finsler connections (1.9) we remark two important.

The first one is the connection obtained from (1.9) if we consider as given connection $F\hat{\Gamma}$ a Matsumoto connection $MF=(\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$ and also $A_j^h=0$, $U_{pk}^q=0$, $V_{pk}^q=0$. We are calling it canonical almost complex connection. Denoted by $F\Gamma=(\overset{q}{N}, \overset{q}{F}, \overset{q}{C})$ such a connection is given by

$$(2.1) \quad \begin{cases} \overset{q}{N}_j^h = \overset{m}{N}_j^h, \\ \overset{q}{F}_{jk}^h = \overset{m}{F}_{jk}^h + \frac{1}{2} \overset{m}{F}_j^p \overset{m}{F}_{p|k}^h, \\ \overset{q}{C}_{jk}^h = \overset{m}{C}_{jk}^h + \frac{1}{2} \overset{m}{F}_j^p \overset{m}{F}_p^h|_k, \end{cases}$$

where $M\Gamma=(\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$ is Matsumoto connection

$$(2.2) \quad \begin{cases} \overset{m}{N}_j^h = \bar{N}_j^h, \\ \overset{m}{F}_{jk}^h = \frac{\partial \bar{N}_j^h}{\partial y^k}, \\ \overset{m}{C}_{jk}^h = \bar{C}_{jk}^h. \end{cases}$$

Here $\bar{N}_j^h$ is a fixed nonlinear connection and $\overset{m}{\cdot}$ means the h -derivative and v -derivative in respect to this connection.

The second important almost complex connection can be obtained from (1.9) for $A_j^h=0$, $U_{pk}^q=0$, $V_{pk}^q=0$. Since it generalizes Kawaguchi connection in the almost complex case it will be called Kawaguchi almost connection associated to Finsler connection $F\hat{\Gamma}$. If denoted $K\hat{\Gamma}=(\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$, we have

$$(2.3) \quad \begin{cases} \overset{k}{N}_j^h = \overset{\cdot}{N}_j^h, \\ \overset{k}{F}_{jk}^h = \overset{\cdot}{F}_{jk}^h + \frac{1}{2} \overset{\cdot}{F}_j^p \overset{\cdot}{F}_{p|k}^h, \\ \overset{k}{C}_{jk}^h = \overset{\cdot}{C}_{jk}^h + \frac{1}{2} \overset{\cdot}{F}_j^p \overset{\cdot}{F}_p^h|_k. \end{cases}$$

We now turn to the problem of obtaining all Finsler connections $F\hat{\Gamma}$ with the property that Kawaguchi almost complex connection $K\hat{\Gamma}$ coincides with the canonical almost complex connection $F\Gamma$. We can prove the

Theorem 2.1. *All Finsler connections $F\hat{\Gamma}$ with the property $K\hat{\Gamma}=F\Gamma$ are given by*

$$(2.4) \quad \begin{cases} \overset{q}{N}_j^h = \overset{q}{N}_j^h = \overset{m}{N}_j^h, \\ \overset{q}{F}_{jk}^h = \overset{q}{F}_{jk}^h + \overset{q}{C}_{js}^h A_k^s + O^{shp}_{sj} X_{pk}^s = \overset{m}{F}_{jk}^h + \frac{1}{2} \overset{m}{F}_j^p \overset{m}{F}_{p|k}^h + O^{shp}_{sj} X_{pk}^s, \\ \overset{q}{C}_{jk}^h = \overset{q}{C}_{jk}^h + O^{shp}_{sj} Y_{pk}^s = \overset{m}{C}_{jk}^h + \frac{1}{2} \overset{m}{F}_j^p \overset{m}{F}_p^h|_k + O^{shp}_{sj} Y_{pk}^s, \end{cases}$$

where $MT = (N, \overset{m}{F}, \overset{m}{C})$ is a Matsumoto connection and X_{pk}^s, Y_{pk}^s are arbitrary fields of Finsler tensors.

Proof. Let be $A_j^h, B_{jk}^h, D_{jk}^h$ the tensors obtained as difference of the pair $(F\overset{q}{\Gamma}, F\overset{q}{\Gamma})$:

$$(2.5) \quad \begin{cases} N_j^h = N_j^h - A_j^h, \\ \overset{q}{F}_{jk}^h = \overset{q}{F}_{jk}^h + \overset{q}{C}_{js}^h A_k^s - B_{jk}^h, \\ \overset{q}{C}_{jk}^h = \overset{q}{C}_{jk}^h - D_{jk}^h. \end{cases}$$

Taking account of (2.5) in (2.3) and also that the covariant h -derivative and covariant v -derivative with respect to the $F\overset{q}{\Gamma}$ connection are equal to zero, we get

$$(2.6) \quad \begin{cases} N_j^h = N_j^h - A_j^h, \\ \overset{k}{F}_{jk}^h = \overset{q}{F}_{jk}^h + \overset{q}{C}_{js}^h A_k^s + \frac{1}{2} F_j^p A_p^s F_{p|s}^{hm} - O_{js}^{ph} B_{pk}^s, \\ \overset{k}{C}_{jk}^h = \overset{q}{C}_{jk}^h - O_{js}^{ph} D_{pk}^s. \end{cases}$$

By imposing the condition $K\overset{q}{\Gamma} = F\overset{q}{\Gamma}$, the last equations become

$$(2.7) \quad A_j^h = 0, \quad O_{jm}^{ph} B_{pk}^m = 0, \quad O_{jm}^{ph} D_{pk}^m = 0.$$

The general solution of system (2.7) is

$$(2.8) \quad A_j^h = 0, \quad B_{jk}^i = O_{hj}^{*is} X_{sk}^h, \quad D_{jk}^i = O_{hj}^{*is} Y_{sk}^h,$$

where X_{sk}^h and Y_{sk}^h are arbitrary tensor fields. If we introduce (2.8) into (2.5) and take account of (2.1), we get (2.4).

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ASUPRA UNEI CLASE DE CONEXIUNI FINSLER

(Rezumat)

Se stabilește o nouă formă a formulelor care exprimă conexiuni Finsler compatibile cu o structură aproape complexă și se determină clasa de conexiuni Finsler cu proprietatea că conexiunea aproape complexă Kawaguchi coincide cu conexiunea aproape complexă canonică a structurii Finsler respective.

CA-COLLINEATION IN A GENERALIZED BIREFRECTION FINSLER MANIFOLD

BY

R.S.D. DUBEY and AJAI KUMAR SRIVASTAVA

1. An n -dimensional Finsler manifold is called generalized birecurrent of first kind [1] if

$$(1.1) \quad H_{hjk(l)(m)}^i = a_{lm} H_{hjk}^i + b_m H_{hjk(l)}^i$$

and of the second kind if

$$(1.2) \quad H_{hjk(l)(m)}^i = a_{lm} H_{hjk}^i + b_l H_{hjk(m)}^i.$$

Here H_{hjk}^i is Berwald's curvature tensor. Non zero tensor a_{ij} and non zero vector b_i are called respectively associate tensor and vector of recurrence. An index in () denotes covariant derivative in the sense of Berwald [2].

It is easy to see that above two kinds of generalized birecurrent manifolds reduce to birecurrent one if the curvature tensor H_{hjk}^i is recurrent.

An infinitesimal coordinate transformation

$$(1.3) \quad \bar{x}^i = x^i + v^i(x)\epsilon,$$

where ϵ is an infinitesimal constant, is called contra-transformation if the vector v^i appearing in (1.3) satisfies

$$(1.4) \quad v_{(j)}^i = \rho \delta_j^i \quad \text{with } \rho \neq 0.$$

If the Lie-derivative of H_{hjk}^i vanishes i.e.

$$(1.5) \quad \mathcal{L} H_{hjk}^i = 0,$$

the transformation (1.3) is called CA-collineation or special CA-collineation according as ρ is function of coordinates or a constant [3].

Using (1.4) in the L.H.S. of (1.5) we get

$$(1.6) \quad v^r H_{hjk(r)}^i + 2\rho H_{hjk}^i = 0.$$

2. In this Note we shall establish two theorems.

Theorem 2.1. *Generalized birecurrent Finsler manifolds do not admit special CA-collineation.*

P r o o f. Let the generalized birecurrent Finsler manifold characterized by (1.1) admit a special CA-collineation. Then integrability condition of (1.4) yields

$$(2.1) \quad v^h H_{hjk}^i = 0,$$

since $\rho = \text{constant}$. Differentiating (2.1) covariantly and using (1.4) we get

$$(2.2) \quad v^h H_{hjk(l)}^i + \rho H_{ljk}^i = 0.$$

Differentiating (2.2) covariantly and using (1.1), (2.1) and (2.2) we get

$$(2.3) \quad H_{ljk(m)}^i + H_{mjk(l)}^i = b_m H_{ljk}^i$$

as $\rho \neq 0$. The L.H.S. is symmetric in l and m . Hence we must have $b_m H_{ljk}^i = b_l H_{mjk}^i$. Transvecting this with v^m we get, in view of (2.1),

$$(2.4) \quad v^m b_m = 0,$$

as $H_{ljk}^i \neq 0$.

Also, transvecting (2.3) with v^m and using (1.6) and (2.2) we get

$$(2.5) \quad v^m b_m + 3\rho = 0.$$

From (2.4) and (2.5) we get $\rho = 0$, a contradiction.

A similar proof holds for the manifold characterized by (1.2).

Theorem 2.2. Every generalized birecurrent Finsler manifold admitting CA-collineation is recurrent (hence birecurrent).

P r o o f. Let the generalized birecurrent Finsler manifold characterized by (1.1) admit CA-collineation. The relation (1.6) still holds in view of (1.3), (1.4) and (1.5). Differentiating (1.6) covariantly and using (1.1) and (1.6) we get

$$(2.6) \quad H_{hjk(m)}^i = K_m H_{hjk}^i,$$

where

$$(2.7) \quad K_m = -\frac{1}{3\rho} (v^r a_{rm} - 2\rho b_m + 2\rho_m)$$

and $\rho_m = \rho_{(m)}$. Transvecting (2.6) by v^m and using (1.6) and (2.7) we get

$$(2.8) \quad 6\rho^2 = a_{rm} v^r v^m - 2\rho b_r v^r + 2\rho_r v^r.$$

Transvecting (1.6) with $v^j v^h$ and contracting i and k we get

$$(2.9) \quad \rho = -\frac{1}{2H} v^r H_{(r)},$$

where H is defined in [2, p. 129].

Note 1. In case of manifold characterized by (1.2) we still get (2.6) with value of K_m now given by

$$(2.10) \quad K_m = -\frac{v^r a_{rm} + 2\rho_m}{3\rho + v^r b_r}.$$

The equation (2.7) is unaltered.

Note 2. As remarked earlier the generalized birecurrent Finsler manifolds reduce to birecurrent one if they are recurrent. Hence generalized birecurrent Finsler manifold is birecurrent if it admits a CA-collineation, according to Theorem 2.

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CA-COLINIAȚIE ÎNTR-O VARIETATE FINSLER BIRECURRENTĂ GENERALIZATĂ

(Rezumat)

Se demonstrează că o varietate Finsler birecurrentă generalizată nu admite CA-coliniație specială iar dacă un astfel de spațiu admite CA-coliniație, atunci el este recurrent.

ÉQUATIONS AUX DÉRIVÉES PARTIELLES, ANALYTIQUES DANS
DES ESPACES DE BANACH

PAR

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0. Introduction. Dans ce travail on considère des équations aux dérivées partielles, analytiques dans des espaces de Banach, qui généralisent les systèmes de Cauchy-Kowalewsky, la dérivée utilisée étant la dérivée Fréchet. La notion de fonction analytique y est considérée dans un sens suggéré par la définition de H. Cartan [1] d'un polynôme homogène de degré n .

Cette idée de fonction analytique et son application à l'étude de ces problèmes appartiennent à M. le professeur A. Haimovici, qu'à cette occasion je veux remercier une fois de plus.

1. Polynômes homogènes et fonctions analytiques dans des espaces de Banach. *Polynômes homogènes.* Définition [1, Ch. 1.6]. Soit E et F des espaces de Banach. La fonction $\varphi_n: E \rightarrow F$ est nommée polynôme homogène de degré n , s'il existe une application n -linéaire $f_n: E^n \rightarrow F$ ainsi que $\varphi_n(x) = f_n(x, x, \dots, x)$, $\forall x \in E$.

Les fonctions $\varphi_0: E \rightarrow F$, constantes, sont par définition des polynômes de degré 0.

Remarques 1. Si $\varphi_n(x) = f_n(x, \dots, x)$ est un polynôme de degré n , alors $g_n: E^n \rightarrow F$ où $g_n(x_1, x_2, \dots, x_n) = \frac{1}{n!} \sum_{(i_1, i_2, \dots, i_n)} f_n(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ est une forme n -

linéaire symétrique et $\varphi_n(x) = g_n(x, x, \dots, x)$. Si nous notons $(\Delta_h \varphi_n) = \varphi(x+h) - \varphi(x)$ et si $\varphi_n: E \rightarrow F$ est un polynôme homogène de degré n , défini par la forme n -linéaire symétrique $f_n: E^n \rightarrow F$, alors $f_n(x_1, x_2, \dots, x_n) = \Delta_{x_1} \Delta_{x_2} \dots \Delta_{x_n} \varphi_n$ [1].

Ainsi, un polynôme homogène de degré n peut être toujours représenté, uniquement par une forme n -linéaire symétrique. En ce qui suit nous noterons par la même lettre le polynôme et sa forme n -linéaire symétrique correspondante.

2. Les dérivées Fréchet d'un polynôme homogène. Soient $\varphi_n: E \rightarrow F$ un polynôme homogène de degré n et supposons que la forme n -linéaire symétrique $\varphi_n: E^n \rightarrow F$ est continue. Alors le polynôme φ_n a des dérivées Fréchet, d'un ordre quelconque

$$\varphi'_n(x)(h) = n \varphi_n(x, x, \dots, x, h)$$

.....
n-1 fois

$$(\varphi_n^k(x))(h_1, h_2, \dots, h_k) = n(n-1)\dots(n-k+1)\varphi_n(x, \dots, x, h_1, \dots, h_k),$$

$n-k \text{ fois}$

$$(\varphi_n^{(n)}(x))(h_1, h_2, \dots, h_n) = n! \varphi_n(h_1, h_2, \dots, h_n),$$

$$\varphi_n^{(n+k)}(x) \equiv 0, \quad k=1, 2, \dots$$

On notera

$$\varphi_n(x, \dots, x; h_1, h_2, \dots, h_k) = \varphi_n(x^{n-k}, h_1, h_2, \dots, h_k).$$

$n-k \text{ fois}$

Si $u: G \rightarrow E$, G aussi un espace de Banach, est différentiable Fréchet, alors $(\varphi_n(u(x)))'(h) = n\varphi_n[u(x)]^{n-1}; u'(x)(h)[1]$.

Fonctions analytiques. Soit $\varphi: U \rightarrow F$, $\varphi(x) = \sum_{n=0, \infty} \varphi_n(x)$, $\forall x \in E$, où

$\varphi_n(x)$, $n=0, \infty$ sont des polynômes homogènes correspondants à des formes linéaires symétrique continues $\varphi_n: F^n \rightarrow F$, et U est un voisinage de l'origine de E , $U = \{x/\|x\| < r, r > 0\}$. Si $\sum_{n=0, \infty} \|\varphi_n\| \|x\|^n < \infty$ pour $\|x\| < r$ on

dit que φ est analytique dans le voisinage U de $0 \in E$.

Si φ est une fonction analytique dans un voisinage U de l'origine de E , alors il existe

$$\varphi^{(k)}(x)(h_1, h_2, \dots, h_k) = \sum_{n=k, \infty} \varphi_n^{(k)}(x^{n-k}; h_1, h_2, \dots, h_k) \text{ et } \varphi_0 = \varphi(0), \varphi_1(x) = \varphi'(0)(x)$$

$$\varphi_2(x) = \frac{1}{2} \varphi''(0)(x^2) \dots \varphi_n(x) = \frac{1}{n!} \varphi^{(n)}(0)(x^n);$$

([1], 3.6.1) donc, si $f(x)$ et $g(x)$ sont analytiques dans un voisinage de l'origine, $U \subset E$,

$$f(x) = \sum_{n=0, \infty} f_n(x), \quad g(x) = \sum_{n=0, \infty} g_n(x) \text{ et } f(x) = g(x) \quad \forall x \in U,$$

alors $f_n(x) = g_n(x)$; $n=0, 1, 2, \dots$

Fonctions analytiques de plusieurs variables. Soit $E_1, E_2, \dots, E_k, F$ des espaces de Banach et dans les espaces E_i , $i=\overline{1, k}$ soit les boules $B_i = \{x/\|x\| < \varepsilon, \varepsilon > 0\}$.

Une fonction $f: B_1 \times B_2 \times \dots \times B_k \rightarrow F$ avec

$$f(x_1, x_2, \dots, x_k) = \sum_{m \in N_*^k} f_m(x), \quad x = (x_1, x_2, \dots, x_k), \quad x_i \in B_i,$$

$$N_* = \{0, 1, 2, \dots\}, \quad m = (m_1, m_2, \dots, m_k),$$

où $f_m: E_1 \times E_2 \times \dots \times E_k \rightarrow F$ est un polynôme homogène de degré m_i , $i=\overline{1, k}$ par rapport à la variable x_i , ainsi que $(x_1^1, \dots, x_1^{m_1}; x_2^1, \dots, x_2^{m_2}; \dots; x_k^1, \dots, x_k^{m_k}) \rightarrow f_m(x_1^1, \dots, x_1^{m_1}; x_2^1, \dots, x_2^{m_2}; \dots; x_k^1, \dots, x_k^{m_k}) \in F$ est une forme $m_1 + m_2 + \dots + m_k$ -linéaire et continue et la série $\sum_{m \in N_*^k} \|f_m\| \|x_1\|^{m_1} \|x_2\|^{m_2} \dots \|x_k\|^{m_k} < \infty$

pour $\|x_i\| < \varepsilon$, $i=\overline{1, k}$ est nommée analytique dans le voisinage $B_1 \times B_2 \times \dots \times B_k$.

Remarque. Si f et g sont analytiques dans le voisinage

$$B_1 \times B_2 \times \dots \times B_k \quad \text{et} \quad f(x) = \sum_{m \in N_*^k} f_m(x) \quad \text{et} \quad g(x) = \sum_{m \in N_*^k} g_m(x),$$

alors $f_m(x) = g_m(x)$, $\forall m \in N_*^k$.

Fonctions analytiques composées. Proposition. Soit E, F, G , des espaces de Banach.

Si

$$f: V \rightarrow G \quad \text{et} \quad g: U \rightarrow V, \quad U \subset E, \quad U = \{x/x \in E, \|x\| < \varepsilon, \varepsilon > 0\},$$

$$V \subset F, \quad V = \{u/u \in F, \|u\| < \delta, \delta > 0\}$$

sont des fonctions analytiques, alors il existe $h(x) = f(g(x))$, $x \in \tilde{U} \subseteq U$ et h est une fonction analytique.

Démonstration. Puisque g et f sont analytiques on a :

$$g(x) = \sum_{n=0, \infty} g_n(x) \quad \text{avec} \quad \sum_{n=0, \infty} \|g_n\| \|x\|^n < \infty \quad \text{pour} \quad \|x\| < \varepsilon \quad \text{et} \quad f(x) = \sum_{m=0, \infty} f_m(x)$$

$$\text{avec} \quad \sum_{m=0, \infty} \|f_m\| \|u\|^m < \infty \quad \text{pour} \quad \|u\| < \delta.$$

Alors, pour $\|x\| < \tilde{\varepsilon} \leq \varepsilon$, $\sum_{n=0, \infty} \|g_n\| \|x\|^n < \delta$. Donc

$$^*) \quad \sum_{m=0, \infty} \|f_m\| \left(\sum_{n=0, \infty} \|g_n\| \|x\|^n \right)^m < \infty.$$

Dans cette série on peut considérer les termes dans un ordre arbitraire.

Pour $\|x\| < \tilde{\varepsilon}$ on a aussi

$$^{**}) \quad f(g(x)) = \sum_{n=0, \infty} f_n \left(\sum_{m=0, \infty} g_m(x) \right) = \sum_{n=0, \infty} \left(\sum_{i_1, i_2, \dots, i_n=0, \infty} f_n(g_{i_1} g_{i_2} \dots g_{i_n}) \right)$$

parce que $f_n, n=0, \infty$ sont continues. En écrivant formellement ces termes dans un autre ordre, on a

$$\begin{aligned} f(g(x)) = & f_0(g_0) + f_1(g_0) + \dots + f_n(g_0) + \dots + f_1(g_1) + 2f_2(g_1, g_0) + \\ & + 3f_3(g_1, g_0^2) + \dots + f_1(g_2) + f_2(g_1) + 2f_2(g_2, g_0) + 3f_3(g_1^2, g_0) + 3f_3(g_2, g_0^2) + \\ & + \dots = h_0 + h_1(x) + h_2(x) + \dots + h_n(x) + \dots, \end{aligned}$$

où

$$h_0 = f_0(g_0) + f_1(g_0) + \dots + f_n(g_0) + \dots,$$

$$h_1(x) = f_1(g_1(x)) + 2f_2(g_1(x), g_0) + 3f_3(g_1(x), g_0^2) + \dots,$$

$$h_2(x) = f_1(g_2(x)) + f_2(g_1(x)) + 2f_2(g_2(x), g_0) + 3f_3(g_2, g_0^2) + 3f_3(g_1^2, g_0) + \dots$$

.....

Parce que $\|g_0\| < \delta$ et $\sum_{n=0, \infty} \|f_n\| \|u\|^n < \infty$ pour $\|u\| < \delta$, on a

$$\|h_0\| \leq \sum_{n=0, \infty} \|f_n\| \|g_0\|^n < \infty,$$

$$\|h_1\| \leq \left(\sum_{n=1, \infty} n \|f_n\| \|g_0\|^{n-1} \right) \|g_1\| < \infty,$$

$$\|h_1\| \leq \left(\sum_{n=1, \infty} C_n^1 \|f_n\| \|g_0\|^{n-1} \right) \|g_2\| + \left(\sum_{n=1, \infty} C_n^2 \|f_n\| \|g_0\|^{n-2} \right) \|g_1\|^2 < \infty.$$

Donc $h_0, h_1, h_2, \dots$ sont des polynômes homogènes continus. À cause de $(*)$, $\sum_{n=0, \infty} \|h_n\| \|x\|^n < \infty$ pour $\|x\| < \tilde{\varepsilon}$. D'où il résulte que le changement de l'ordre de sommation dans $(**)$ est permis, donc $f(x) = \sum_{n=0, \infty} h_n(x)$ avec $\sum_{n=0, \infty} \|h_n\| \|x\|^n < \infty$ pour $\|x\| < \tilde{\varepsilon}$, c'est-à-dire f est une fonction analytique.

3. Équations aux dérivées partielles, analytiques, dans des espaces de Banach. Soit les espaces de Banach E_0, E, F et les boules

$B_0 = \{t/t \in E_0, \|t\| < \varepsilon_0\}$, $B = \{x/x \in E_1, \|x\| < \varepsilon_0\}$, $G = \{u/u \in F, \|u\| < \varepsilon_0\}$. Soit

$$f: B_0 \times B \times G \times G \rightarrow L(E_0, F),$$

$$f(t, x, u, v) = \sum_{m, n, p, q=0, \infty} A_{mnpq}(t, x, u, v)$$

où $A_{mnpq}: E_0^m \times E_0^n \times F^p \times F^q \rightarrow L(E_0, F)$ est $m+n+p+q$ -linéaire et continue,

$$(I_1) \quad A_{mnpq}(t_1, t_2, \dots, t_m; x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_p; v_1, v_2, \dots, v_q) \in L(E_0, F)$$

étant symétrique par rapport aux variables t, x, u, v respectivement, c'est-à-dire, $A_{mnpq}: E_0 \times E \times F \times F \rightarrow L(E_0, F)$ avec $A_{mnpq}(t, x, u, v) = A_{mnpq}(\overbrace{t, \dots, t}^{m \text{ fois}}, \overbrace{x, \dots, x}^{n \text{ fois}}, \overbrace{u, \dots, u}^{p \text{ fois}}, \overbrace{v, \dots, v}^{q \text{ fois}})$ est un polynôme homogène continu de degré m, n, p, q , en t, x, u, v . Supposons aussi que

$$(I_2) \quad \sum_{m, n, p, q=0, \infty} \|A_{mnpq}\| \|t\|^m \|x\|^n \|u\|^p \|v\|^q < \infty$$

pour $\|t\| < \varepsilon_0, \|x\| < \varepsilon_0, \|u\| < \varepsilon_0, \|v\| < \varepsilon_0$.

À cause de (I_1) et (I_2) on voit que f est une fonction analytique dans le voisinage $B_0 \times B \times G \times G \subset E_0 \times E \times F \times F$. Parce qu'une fonction analytique d'une fonction analytique est toujours une fonction analytique il est justifié de chercher une fonction analytique $u = u(t, x) \in F$ pour $t \in E_0, x \in E, \|t\| < \varepsilon_0, \|x\| < \varepsilon_0$, ainsi que

$$(1) \quad \frac{\partial u(t, x)}{\partial t} (h_1) = f\left(t, x, u(t, x), \frac{\partial u(t, x)}{\partial x} (x_0)\right) (h_1),$$

avec la condition

$$(2) \quad u(0, x) = 0,$$

où f satisfait les conditions $(I_1), (I_2)$, h_1 étant arbitraire, x_0 étant aussi arbitraire mais fixé.

On va démontrer le

Théorème 1. Si f satisfait les conditions $(I_1), (I_2)$, alors l'équation (1) avec la condition (2) admet une seule solution analytique dans un voisinage de l'origine de $E_0 \times E$.

Démonstration. Nous allons montrer d'abord l'unicité de la solution. Soit

$$(I_1) \quad u(t, x) = \sum_{\substack{\alpha=0, \infty \\ \beta=0, \infty}} u_{\alpha\beta}(t, x)$$

une solution analytique du problème (1), (2) c'est-à-dire, $u_{\alpha\beta} = u_{\alpha\beta}(t_1, t_2, \dots, t_\alpha, x_1, x_2, \dots, x_\beta) \in F$ est $(\alpha + \beta)$ -linéaire, continue et symétrique par rapport aux variables $t_i \in E_0$ et $x_i \in E$, donc $u_{\alpha\beta}(t, x) = u_{\alpha\beta}(\underbrace{t, \dots, t}_{\alpha \text{ fois}}, \underbrace{x, \dots, x}_{\beta \text{ fois}})$, $t \in E_0, x \in E$ est un polynôme homogène de degré α en t et β en x ainsi que

$$(I_2) \quad \sum_{\alpha, \beta=0, \infty} \|u_{\alpha\beta}\| \|t\|^\alpha \|x\|^\beta < \infty \quad \text{pour} \quad \|x\| < \delta_0, \|t\| < \delta_0, \delta_0 > 0.$$

À cause de la condition (2) il s'en suit nécessairement $\sum_{\beta=0, \infty} u_{0\beta}(t, x) = 0$ d'où $u_{0\beta}(t, x) = 0, \beta = \overline{0, \infty}$ et donc

$$(I_1'') \quad u(t, x) = \sum_{\substack{\alpha=1, \infty \\ \beta=0, \infty}} u_{\alpha\beta}(t, x).$$

En remplaçant dans l'équation (1) $t=0$ on obtient, tenant compte de

$$(I_1') \quad \text{et} \quad (I_1) \quad \sum_{\beta=0, \infty} u_{1\beta}(h_1, x) = \sum_{m, n, p, q=0, \infty} A(0, x, 0, 0)(h_1) = \sum_{n=0, \infty} A_{0n00}(0, x, 0, 0)(h_1).$$

Ayant en vue l'analyticit  suppos e, on peut d river terme   terme d'un nombre quelconque de fois les s ries qui apparaissent ici. Ainsi, pour obtenir les polyn mes $u_{i\beta}$, nous, d rivons par rapport   t l' quation (1) et puis nous rempla ons $t=0$. On obtient d'abord

$$\begin{aligned} \frac{\partial^2 u(t, x)}{\partial t^2}(h_1, h_2) &= \sum_{\substack{\alpha=1, \infty \\ \beta=0, \infty}} \alpha(\alpha-1) u_{\alpha\beta}(t^{x-2}, h_1, h_2; x) = \\ &= \sum_{m, n, p, q} m A_{mnpq} \left(t^{m-1}, h_2; x, u(t, x), \frac{\partial u(t, x)}{\partial x}(x_0) \right) (h_1) + \\ &+ \sum_{m, n, p, q} p A_{mnpq} \left(t, x; (u(t, x))^{p-1}, \sum_{\substack{i=1, \infty \\ j=0, \infty}} i u_{ij} \left((t^{i-1}, h_2; x); \frac{\partial u(t, x)}{\partial x}(x_0) \right) (h_1) + \right. \\ &+ \left. \sum_{m, n, p, q} q A_{mnpq} \left(t, x, u(t, x); \left(\frac{\partial u(t, x)}{\partial x}(x_0) \right)^{q-1}, \sum_{\substack{i=1, \infty \\ j=1, \infty}} i j u_{ij} (t^{i-1}, h_2; x^{j-1}, x_0) (h_1) \right) \right) \end{aligned}$$

et puis

$$\begin{aligned} \sum_{\beta=0, \infty} 2u_{2\beta}(h_1, h_2; x) &= \sum_{n=0, \infty} A_{1n00}(h_2, x, 0, 0) + \\ &+ \sum_{n=0, \infty} A_{0n10}(0, x, \sum_{j=0, \infty} u_{1j}(h_2, x), 0) (h_1) + \\ &+ \sum_{n=0, \infty} A_{0n01}(0, x, \sum_{j=1, \infty} j u_{1j}(h_2; x^{j-1}, x_0) (h_1) \end{aligned}$$

et ainsi

$$(3) \quad 2u_{2\beta}(h_1, h_2, x) = A_{1\beta 00}(h_2, x)(h_1) + \sum_{n+j=\beta} A_{0n10}(x, u_{1j}(h_2, x)(h_1) + \\ + \sum_{n+j=\beta+1} j A_{0n01}(x; u_{1j}(h_2; x^{j-1}, x_0))(h_1).$$

Les formes linéaires qui déterminent les polynômes homogènes $u_{2\beta}$ dans (3) se rendent symétriques par rapport aux variables t et x respectivement et en cette forme s'utilise pour déterminer les polynômes $u_{3\beta}$, $\beta = \overline{0, \infty}$. Avec $u_{3\beta}, \dots, u_{\alpha\beta}$, on va procéder de la même manière.

Les polynômes $u_{3\beta}, \dots, u_{\alpha\beta}, \dots$ se déterminent de manière analogue des égalités

$$\frac{\partial^3 u(t, x)}{\partial t^3} (h_1, h_2, h_3) = \frac{\partial^3 f}{\partial t^3} (h_1, h_2, h_3),$$

.....

$$\frac{\partial^\alpha u(t, x)}{\partial t^\alpha} (h_1, h_2, \dots, h_\alpha) = \frac{\partial^{\alpha-1} f}{\partial t^{\alpha-1}} (h_1, h_2, \dots, h_\alpha),$$

.....

où l'on remplace $t=0$, tenant compte des résultats antérieurs. De cette manière est démontrée l'unicité de la solution analytique du problème (1), (2).

Pour démontrer l'existence de la solution analytique du problème (1), (2), on attache à ce problème une équation analytique scalaire aux dérivées partielles dont la solution sera un majorant pour la série de (I_2') où $\sum u_{\alpha\beta}$ est celle déterminée antérieurement. Cette équation est

$$(1') \quad \frac{\partial \bar{U}(\bar{t}, \bar{x})}{\partial \bar{t}} = F\left(\bar{t}, \bar{x}, \bar{U}(\bar{t}, \bar{x}), \frac{\partial \bar{U}(\bar{t}, \bar{x})}{\partial \bar{t}} \|x_0\|\right),$$

avec la condition

$$(2') \quad \bar{U}(0, \bar{x}) = 0 \text{ où } F(\bar{t}, \bar{x}, \bar{u}, \bar{v}) = \sum \|A_{mnpq}\| \bar{t}^m \bar{x}^n \bar{u}^p \bar{v}^q$$

pour $(\bar{t}, \bar{x}, \bar{u}, \bar{v}) \in R^4$ avec $|\bar{t}|, |\bar{x}|, |\bar{u}|, |\bar{v}| < \varepsilon_0$.

L'équation (1') avec la condition (2') a une seule solution analytique

$$\bar{U}(\bar{t}, \bar{x}) = \sum_{\substack{\alpha=\overline{1, \infty} \\ \beta=\overline{0, \infty}}} \bar{U}_{\alpha\beta} \bar{t}^\alpha \bar{x}^\beta; \quad |\bar{t}|, |\bar{x}| < \delta_0 \text{ où } \bar{U}_{1\beta} = \|A_{0\beta 00}\|, \beta = \overline{0, \infty},$$

$$\sum 2\bar{U}_{2\beta} \bar{x}^\beta = \sum_{n=0, \infty} \|A_{1n00}\| \bar{x}^n + \sum_{n=0, \infty} \|A_{0n10}\| \bar{x}^n \left(\sum_{j=0, \infty} \bar{U}_{1j} \bar{x}^j \right) + \\ + \sum_{n=0, \infty} \|A_{0n01}\| \bar{x}^n \left(\sum_{j=1, \infty} j \bar{U}_{1j} \bar{x}^{j-1} \right) \|x_0\| \text{ c.t.c.}$$

c'est-à-dire

$$(3') \quad \bar{U}_{2\beta} = \|A_{1\beta 00}\| + \sum_{n+j=\beta} \|A_{0n10}\| \bar{U}_{1j} + \sum_{n+j=\beta+1} j \|A_{0n01}\| \bar{U}_{1j} \|x_0\| \text{ e.t.c.}$$

Des expressions trouvées pour $u_{\alpha\beta}$ et $\bar{U}_{\alpha\beta}$, on déduit que $\|u_{\alpha\beta}\| \leq \bar{U}_{\alpha\beta}$ et ainsi la série $u(t, x) = \sum u_{\alpha\beta}(t, x)$ avec $u_{\alpha\beta}$ trouvés antérieurement, a la

propriété que $\sum \|U_{\alpha\beta}\| \|t\|^\alpha \|x\|^\beta < \infty$ pour $\|t\| < \delta_0$, $\|x\| < \delta_0$, c'est-à-dire $u(t, x)$ est analytique dans un voisinage de l'origine de $E_0 \times E_1$. Le théorème est ainsi complètement démontré. Le problème traité peut être généralisé aussi pour des fonctions qui dépendent d'un nombre quelconque, fini, de variables, avec une démonstration analogue.

Pour cela, considérons les espaces de Banach $E_0, E_1, \dots, E_k, F$ et soit dans ces espaces, les boules $B_i = \{x_i \in E_i, \|x_i\| < \varepsilon_0\}$, $i = \overline{0, k}$ et $G = \{x | x \in F, \|x\| < \varepsilon_0\}$. Soit $f : B_0 \times B_1 \dots \times B_k \times G \times G^k \rightarrow L(E_0, F)$, où

$$(I_3) \quad f(t, x_1, x_2, \dots, x_k, u, v_1, v_2, \dots, v_k) = \sum_{m, \alpha, p, \beta} A_{m\alpha p\beta}(t, x, h, v).$$

Ici

$$m \in N_* = \{0, 1, 2, \dots\}, p \in N_*, \alpha = (\alpha_1, \alpha_2, \dots, \alpha_k), \alpha_i \in N_*, \beta = (\beta_1, \beta_2, \dots, \beta_k),$$

$$\beta_i \in N_*, x = (x_1, x_2, \dots, x_k), x_i \in E_i, v = (v_1, v_2, \dots, v_k), v_i \in F,$$

$A_{m\alpha p\beta}$ est $m + (\alpha_1 + \dots + \alpha_k) + p + (\beta_1 + \dots + \beta_k)$ -linéaire et continue et

$$A_{m\alpha p\beta}(t_1, \dots, t_m; x_1^1, \dots, x_1^{\alpha_1}, \dots, x_k^{\alpha_k}, \dots, x_k^{\alpha_k}; u_1, \dots, u_p;$$

$$v_1^1, \dots, v_1^{\beta_1}; \dots, v_k^1, \dots, v_k^{\beta_k}) \in L(E_0, F).$$

est symétrique par rapport aux variables $t_i, x_1^i, \dots, x_k^i, u_i, v_1^i, \dots, v_k^i$, donc $A_{m\alpha p\beta}$ est un polynôme homogène de degré m par rapport à la variable t , de degré $\alpha_1, \alpha_2, \dots, \alpha_k$ par rapport aux $x_1, x_2, \dots, x_k$, de degré p en u et de degré $\beta_1, \beta_2, \dots, \beta_k$ respectivement, par rapport aux variables $v_1, v_2, \dots, v_k$.

Supposons que

$$(I_4) \quad \sum \|A_{m\alpha p\beta}\| \|t\|^m \|x_1\|^{\alpha_1} \dots \|x_k\|^{\alpha_k} \|u\|^p \|v_1\|^{\beta_1} \dots \|v_k\|^{\beta_k} < \infty$$

pour $(t, x_1, \dots, x_k, u, v_1, \dots, v_k)$ avec $\|t\| < \varepsilon_0$, $\|x_i\| < \varepsilon_0$ ($i = \overline{1, k}$), $\|u\| < \varepsilon_0$, $\|v_i\| < \varepsilon_0$ ($i = \overline{1, k}$).

En ce cas est vrai le suivant

Théorème 2. Soit l'équation

$$(4) \quad \frac{\partial u(t, x_1, \dots, x_k)}{\partial t}(h_1) = f\left(t, x_1, x_2, \dots, x_k, u, \frac{\partial u}{\partial x_1}(x_{01}), \frac{\partial u}{\partial x_2}(x_{02}), \dots, \frac{\partial u}{\partial x_k}(x_{0k})\right)(h_1)$$

avec la condition

$$(5) \quad u(0, x_1, \dots, x_k) = 0,$$

où f est une fonction analytique, c'est-à-dire elle satisfait les conditions (I_3) , (I_4) et $x_{0i} \in E_i$, $i = \overline{1, k}$ sont des éléments fixés.

Alors l'équation (4) avec la condition (5) a une seule solution analytique, dans un voisinage de l'origine de $E_0 \times E_1 \dots \times E_k$.

Remarque. Si $E_0 = E_1 = \dots = E_k = R$, $F = R^n$ et $x_{01} = x_{02} = \dots = x_{0k} = 1$, on obtient le théorème Cauchy-Kowalewsky pour les systèmes analytiques réels.

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ECUAȚII CU DERIVATE PARȚIALE ANALITICE ÎN SPAȚII BANACH

(Rezumat)

Se definesc mai întâi funcții analitice în spații Banach cu ajutorul polinoamelor omogene introduse de H. Cartan în [1] și se demonstrează că o funcție compusă cu funcții analitice este o funcție analitică. Se arată apoi că ecuația cu derivate parțiale Fréchet (1) cu condiția (2), respectiv ecuația (3) cu condiția (4), are soluție analitică unică, obținând astfel o generalizare în spații Banach a sistemelor analitice clasice Cauchy-Kowalevski.

LIMITA ȘI CONTINUITATEA ANALITICĂ A MULTIFUNCȚIILOR (I)

DE

AL. SABADÎȘ

1. Introducere. Se pleacă de la ideea că pentru funcțiile obișnuite definite pe un spațiu topologic cu valori în alt spațiu topologic, noțiunea de limită într-un punct poate fi definită cu ajutorul șirurilor sau cu ajutorul vecinătăților, cele două definiții fiind echivalente. Definirea limitei cu ajutorul șirurilor este posibilă datorită faptului că orice spațiu topologic este un spațiu de convergență. În cazul unei multifuncții definite pe X cu valori în Y , (X, Y, F) , se va folosi spațiul părților lui Y înzestrat cu convergența introdusă în [1], iar X se va considera ca fiind un spațiu de convergență.

Se va utiliza noțiunea de convergență după Fréchet, adică se va considera un spațiu X în care s-a definit într-un mod oarecare convergența șirurilor supusă la următoarele trei condiții :

- a) limita unui șir este unică ;
- b) șirul constant $(a, a, \dots)$ converge către a , oricare ar fi $a \in X$;
- c) dacă șirul $(a_n) \cap X$ converge către $a \in X$, atunci orice subșir al său (a_{n_k}) converge către a atunci când $k \rightarrow \infty$ și $n_k \rightarrow \infty$.

Remarcăm că convergența analitică a șirurilor de mulțimi din $\mathcal{P}(Y)$ definită în lucrarea amintită, are aceste trei proprietăți.

Un punct $x_0 \in X$ se numește punct de acumulare pentru mulțimea $A \subset X$ dacă există un șir $(x_n) \subset A$, astfel încît $x_n \rightarrow x_0$.

În [3] se studiază aplicații ale familiei $\mathcal{P}(X)$ în familia $\mathcal{P}(Y)$. Scopul acestei lucrări, în mai multe părți, constă în studiul complet al limitei și continuității analitice a multifuncțiilor. Mai exact, se introduce noțiunea de limită (continuitate) analitică laterală, se dau proprietăți ale limitelor de multifuncții și criterii de existență ale acestora, se introduc algebrele Boole $\mathcal{L}(E, Y, x_0)$ și $\mathcal{O}(E, Y, x_0)$, se definește noțiunea de continuitate analitică uniformă a unei multifuncții pe o mulțime etc.

2. Noțiunea de limită analitică a unei multifuncții într-un punct. În cele ce urmează, se va nota cu X un spațiu de convergență iar cu Y o mulțime oarecare.

Definiția 1. Fie multifuncția (E, Y, F) , unde $E \subset X$, și fie $x_0 \in X$, un punct de acumulare pentru mulțimea E . Se spune că o mulțime $A \subset Y$ este limita analitică a multifuncției F în punctul x_0 , dacă și numai dacă pentru orice șir $(x_n) \subset E$, $x_n \rightarrow x_0$, $x_n \neq x_0$, are loc relația $F(x_n) \xrightarrow{(a)} A$. Se notează $(a) - \lim_{x \rightarrow x_0} F(x) = A$.

Parafrăzînd definiția noțiunii de convergență analitică a șirurilor de mulțimi din Y , putem scrie

$$[(a) - \lim_{x \rightarrow x_0} F(x) = A] \Leftrightarrow [\forall (x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow (a) - \lim F(x_n) = A] \Leftrightarrow [\forall (x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow \exists (D_n) \subset \mathcal{P}(Y), D_n \downarrow \emptyset,$$

astfel încît $F(x_n) \subset D_n, \forall n \geq n_0, n_0 \text{ fixat}]$.

Observații. a) Deoarece $\mathcal{P}(Y)$ este un spațiu de convergență, limita oricărui șir de mulțimi este unică. Așadar, se poate scrie

$$[(a) - \lim_{x \rightarrow x_0} F(x) = A] \Rightarrow [\forall (x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow \exists (a) - \lim F(x_n)],$$

rezultînd că limita tuturor șirurilor de mulțimi $(F(x_n)) \subset (Y)$ este aceeași, independentă de șirul $(x_n) \subset E$.

Într-adevăr, dacă pentru două subșiruri $(x_n^1) \subset E$ și $(x_n^2) \subset E$ convergente la x_0 , am avea $F(x_n^1) \xrightarrow{(a)} A_1$ și $F(x_n^2) \xrightarrow{(a)} A_2$, cu $A_1 \neq A_2$, atunci pentru șirul convergent la x_0 , $(x_1^1, x_1^2, x_2^1, x_2^2, \dots)$, am avea $(F(x_1^1), F(x_1^2), F(x_2^1), F(x_2^2), \dots)$, un șir neconvergent analitic, deoarece are două subșiruri cu limite analitice diferite.

b) Negînd propoziția precedentă tragem concluzia că existența în E a două șiruri (x_n^1) și (x_n^2) , cu limita x_0 , astfel încît șirurile de mulțimi $(F(x_n^1))$ și $(F(x_n^2))$ să aibă limite analitice diferite, implică inexistența limitei analitice a multifuncției F în punctul x_0 .

c) Dacă șirul $(x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0$ și $(F(x_n))$ nu este analitic convergent în $\mathcal{P}(Y)$, atunci nu există $(a) - \lim_{x \rightarrow x_0} F(x)$.

Exemple. a) Fie multifuncția (X, Y, F) definită prin $F(x) = A \subset Y$, oricare ar fi $x \in X$. Oricare ar fi $x_0 \in X$ și oricare ar fi șirul $(x_n) \subset X, x_n \neq x_0, x_n \rightarrow x_0$, avem $(F(x_n)) = (A)$, un șir constant în $\mathcal{P}(Y)$, care este analitic convergent spre A .

Deci $(a) - \lim_{x \rightarrow x_0} F(x) = A$, oricare ar fi $x_0 \in X$.

b) Fie multifuncția (X, X, F) definită prin $F(x) = \{x\} \in \mathcal{P}(X)$, oricare ar fi $x \in X$. Oricare ar fi $x_0 \in X$ și oricare ar fi șirul $(x_n) \subset X, x_n \neq x_0, x_n \rightarrow x_0$, avem $F(x_n) = \{x_n\} \in \mathcal{P}(X)$. Studiăm șirul de mulțimi $(F(x_n)) = (\{x_n\}) = (\{x_1\}, \{x_2\}, \dots, \{x_n\}, \dots)$. În $\mathcal{P}(X)$ el este analitic convergent cu limita vidă. Deci putem conchide că $(a) - \lim_{x \rightarrow x_0} F(x) = \emptyset$.

Definiția 2. Fie mulțimea $E \subset X$ și fie multifuncția (E, Y, F) . Considerăm o mulțime $B \subset E$ și x_0 un punct de acumulare al lui B . Spunem că multifuncția F are limită analitică relativ la submulțimea B în punctul x_0 , dacă pentru orice șir $(x_n) \subset B, x_n \neq x_0, x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$. Notăm $(a) - \lim_{x \rightarrow x_0, x \in B} F(x) = A$.

Propoziția 1. Dacă multifuncția (E, Y, F) are limită analitică în punctul de acumulare al mulțimii $B \subset E, x_0$, atunci are în x_0 aceeași limită, relativ la mulțimea $B \subset E$.

Demonstrație. Pentru orice șir $(x_n) \subset B \subset E, x_n \neq x_0, x_n \rightarrow x_0$, avem $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$. Deci $(a) - \lim_{x \rightarrow x_0, x \in B} F(x) = A$, și propoziția este demonstrată.

Definiția 3. Fie $(X, <)$ o mulțime liniar ordonată și $x_0 \in X$, un punct de acumulare a lui $E \subset X$, precum și a lui $B = \{x | x \in E, x < x_0\}$. Limita analitică a multifuncției (E, Y, F) relativ la mulțimea B , se numește limita analitică la stînga a multifuncției F în punctul x_0 .

Notăm, $(a) - \lim_{\substack{x \rightarrow x_0 \\ x \in B}} F(x) = (a) - \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x)$. Cu alte cuvinte, putem spune că $(a) - \lim_{\substack{x \rightarrow x_0 \\ x \in B}} F(x) = A \in \mathcal{P}(Y)$, este echivalent cu a spune că pentru orice șir $(x_n) \subset B, x_n \neq x_0, x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$, sau că, pentru orice șir $(x_n) \subset E, x_n < x_0, x_n \rightarrow x_0$, șirul de mulțimi $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$.

Propoziția următoare oferă posibilitatea reducerii familiei de șiruri $(x_n) \subset E, x_n < x_0, x_n \rightarrow x_0$ la subfamilia șirurilor crescătoare către x_0 .

Propoziția 2. Există $(a) - \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) = A$, dacă și numai dacă pentru orice șir $(x_n) \subset E$, crescător la x_0 , are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$.

Demonstrație. Dacă $(a) - \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) = A$, atunci pentru orice șiruri $(x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} A$. În particular, ultima relația va avea loc și pentru șirurile $(x_n) \subset E, x_n \nearrow x_0$. Într-un sens, propoziția este demonstrată. Reciproc, dacă șirul $(x_n) \subset E, x_n < x_0, x_n \rightarrow x_0$, nu ar fi crescător, atunci el poate fi rearanjat într-un șir crescător $x'_n \nearrow x_0$. Vom avea $F(x'_n) \xrightarrow{(a)} A$ oricare ar fi ordinea termenilor șirului $(F(x'_n))$. Rearanjînd acest șir sub forma $(F(x_n))$, avem $F(x_n) \xrightarrow{(a)} A$ și cu aceasta propoziția este demonstrată.

Observație. Dacă în șirul crescător (x_n) se consideră termenii ce se repetă o singură dată, se obține un șir strict crescător. Prin urmare, este adevărată și următoarea propoziție.

Propoziția 3. Există $(a) - \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) = A$, dacă și numai dacă pentru orice șir $(x_n) \subset E$, strict crescător cu limita x_0 , are loc $F(x_n) \xrightarrow{(a)} A$.

Definiția 4. Fie $(X, <)$ o mulțime liniar ordonată și $x_0 \in X$, un punct de acumulare a lui $E \subset X$, precum și a lui $B = \{x | x \in E, x_0 < x\}$. Limita analitică a multifuncției (E, Y, F) relativ la mulțimea B se numește limita analitică la dreapta a multifuncției F în punctul x_0 .

Notăm $(a) - \lim_{\substack{x \rightarrow x_0 \\ x \in B}} F(x) = (a) - \lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x)$. Cu alte cuvinte, $(a) - \lim_{\substack{x \rightarrow x_0 \\ x \in B}} F(x) = A \in \mathcal{P}(Y)$ dacă și numai dacă pentru orice șir $(x_n) \subset E, x_0 < x_n, x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$. Prin raționamente analoge celor anterioare, se demonstrează și următoarele două propoziții.

Propoziția 4. Există $(a) - \lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x) = A$, dacă și numai dacă pentru orice șir $(x_n) \subset E$, descrescător la x_0 , are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$.

Propoziția 5. Există $(a) - \lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x) = A$, dacă și numai dacă pentru orice șir $(x_n) \subset E$, strict descrescător către x_0 , are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{P}(Y)$.

Sub ipoteza de liniară ordonare a spațiului X , se poate stabili o relație între limita analitică a unei multifuncții într-un punct și limitele sale analitice laterale.

Propoziția 6. Fie $(X, <)$ o mulțime liniar ordonată, iar $x_0 \in X$, un punct de acumulare al mulțimii $E \subset X$ și al mulțimilor $B_s = \{x \mid x \in E, x < x_0\}$, $B_d = \{x \mid x \in E, x_0 < x\}$.

O multifuncție (E, Y, F) are limită analitică în punctul x_0 , dacă și numai dacă ea are în x_0 limite analitice la stînga, respectiv la dreapta, egale. Are loc relația

$$(a) - \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) = (a) - \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) = (a) - \lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x).$$

Demonstrație. Din existența $(a) - \lim_{x \rightarrow x_0} F(x)$ rezultă imediat existența limitelor analitice laterale egale. Reciproc, presupunem că $(a) - \lim_{x \rightarrow x_0} F(x) = (a) - \lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x) = A \in \mathcal{P}(Y)$.

Așadar, oricare ar fi șirul (x_n) , $x_n < x_0$, $x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} A$ și oricare ar fi șirul (x'_n) , $x_0 < x'_n$, $x'_n \rightarrow x_0$ are loc $F(x'_n) \xrightarrow{(a)} A$. Pe de altă parte, orice șir $(x_n) \subset E$, $x_n \rightarrow x_0$ se poate partiționa în subșiruri monotone. Fie $(\bar{x}_n)$ și $(\tilde{x}_n)$ astfel încît $\bar{x}_n \nearrow x_0$ și $\tilde{x}_n \searrow x_0$. În baza ipotezei, $F(\bar{x}_n) \xrightarrow{(a)} A$ și $F(\tilde{x}_n) \xrightarrow{(a)} A$. Dar șirul de mulțimi $(F(x_n))$ va putea fi partiționat în subșiruri de tipul $(F(\bar{x}_n))$ sau $(F(\tilde{x}_n))$. Deci $(F(x_n))$ are ca singură mulțime de acumulare pe A , care este limita lui analitică. Propoziția este demonstrată.

Definiția 5. Fie spațiul liniar ordonat $(X, <)$ și fie x_0 un punct de acumulare a lui $E \subset X$.

a) Multifuncția (E, Y, F) se numește monoton crescătoare pe E dacă pentru orice $x_1, x_2 \in E$, cu $x_1 < x_2$, $F(x_1) \subseteq F(x_2)$.

b) Multifuncția (E, Y, F) se numește monoton descrescătoare pe E , dacă pentru orice $x_1, x_2 \in E$, cu $x_1 < x_2$, $F(x_1) \supseteq F(x_2)$.

Propoziția 7. Fie $(X, <)$ un spațiu liniar ordonat și x_0 un punct de acumulare al mulțimii $E \subset X$. Dacă multifuncția (E, Y, F) este monotonă pe E , atunci ea are limite analitice laterale în punctul x_0 .

Demonstrație. a) Presupunem că pentru orice $x_1, x_2 \in E$, $x_1 < x_2$ are loc $F(x_1) \subseteq F(x_2)$. Deci, pentru orice șir (x_n) cu $x_1 < x_2 < \dots < x_n < \dots$ și $x_n \nearrow x_0$, vom avea $F(x_1) \subseteq F(x_2) \subseteq \dots \subseteq F(x_n) \subseteq \dots$. În acest caz, există $(a) - \lim_{x \rightarrow x_0} F(x) = \bigcup_{n=1}^{\infty} F(x_n)$. Pentru un șir (x'_n) cu $x'_n \searrow x_0$ vom avea $F(x_1) \supseteq F(x_2) \supseteq \dots \supseteq F(x'_n) \supseteq \dots$. Aici vom avea $(a) - \lim_{x \rightarrow x_0} F(x) = \bigcap_{n=1}^{\infty} F(x'_n)$.

Așadar, pentru o multifuncție F monoton crescătoare pe E , există $(a) - \lim_{x \rightarrow x_0} F(x) = \bigcup_{x < x_0} F(x)$ unde $(x_n) \subset E$ este un șir strict crescător la x_0 și $(a) - \lim_{x \rightarrow x_0} F(x) = \bigcap_{x < x_0} F(x)$, unde $(x'_n) \subset E$ este un șir strict descrescător la x_0 .

b) Presupunem că F este descrescătoare pe E . Dacă $x_1 < x_2 < \dots < x_n < \dots$ și $x_n \nearrow x_0$, atunci $F(x_1) \supseteq F(x_2) \supseteq \dots \supseteq F(x_n) \supseteq \dots$ este un șir de mulțimi pentru

care $(a)\text{-}\lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) = \bigcap_1^{\infty} F(x_n)$. Pentru un șir (x'_n) cu $x'_n \searrow x_0$, vom avea $F(x_1) \subseteq \dots \subseteq F(x_2) \subseteq \dots \subseteq F(x_n) \subseteq \dots$ și prin urmare, $(a)\text{-}\lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x) = \bigcup_1^{\infty} F(x'_n)$.

Cu acestea teorema este demonstrată.

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LA LIMITE ET LA CONTINUITÉ ANALITIQUE DES MULTI-APPLICATIONS (I)

(Résumé)

On définit les notions de „limite analitique” et de „limite analitique latérale” (à gauche ou à droite) pour une multi-application dans un point et on établit certaines liaisons entre eux

can (a) $\lim_{t \rightarrow \infty} V(t) = \bigcap_{t \in T} V(t)$, where $\lim_{t \rightarrow \infty} V(t) = \bigcap_{t \in T} V(t)$ is the limit of the sequence $V(t)$ as $t \rightarrow \infty$.
 (b) $\lim_{t \rightarrow \infty} V(t) = \bigcap_{t \in T} V(t)$, where $\lim_{t \rightarrow \infty} V(t) = \bigcap_{t \in T} V(t)$ is the limit of the sequence $V(t)$ as $t \rightarrow \infty$.

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LA LIMITE ET LA CONTINUITE ANALYTIQUE DES MULTISERIES

(Résumé)

On définit les notions de "limites analytiques" et de "limites multiserielles analytiques" et on établit certaines propriétés de ces notions.

GREEN'S FUNCTION FOR A TWO-PARAMETER EVOLUTION SYSTEM

BY

N. GRĂDINARU

We consider the problem of determining the solution of the system

$$(1) \quad \begin{cases} u_{x_1} - \alpha^2 u_{z_1 z_1} = 0, \\ u_{x_1} - \beta^2 u_{z_2 z_2} = 0, \end{cases} \quad \begin{cases} 0 < x_1 \leq T, & 0 < z_2 \leq T, \\ 0 < x_1 \leq T, & 0 < z_2 < b, \end{cases}$$

with the conditions

$$(2) \quad \begin{cases} u(x_1, x_2, 0, z_2) = 0, \\ u(x_1, x_2, a, z_2) = 0, \\ u(x_1, x_2, z_1, 0) = 0, \\ u(x_1, x_2, z_1, b) = 0, \end{cases} \quad \begin{cases} 0 \leq x_1 \leq T, & 0 \leq x_2 \leq T, \\ 0 \leq z_1 \leq a, & 0 \leq z_2 \leq b; \end{cases}$$

$$(3) \quad u(0, 0, z_1, z_2) = \varphi(z_1, z_2).$$

The solution of this problem is [1]

$$u(x_1, x_2, z_1, z_2) = \sum_{n,m=1}^{\infty} \varphi_{nm} \exp \left[-\alpha^2 \left(\frac{n\pi}{a} \right)^2 x_1 - \beta^2 \left(\frac{m\pi}{b} \right)^2 x_2 \right] \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2,$$

where

$$\varphi_{nm} = \frac{4}{ab} \int_0^a \int_0^b \varphi(s, t) \sin \frac{n\pi}{a} s \sin \frac{m\pi}{b} t \, ds \, dt.$$

The series

$$(4) \quad \sum_{n,m=1}^{\infty} \exp \left[-\alpha^2 \left(\frac{n\pi}{a} \right)^2 x_1 - \beta^2 \left(\frac{m\pi}{b} \right)^2 x_2 \right] \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 \sin \frac{n\pi}{a} s \sin \frac{m\pi}{b} t$$

is dominated by the convergent series

$$\sum_{n,m=1}^{\infty} \exp \left[-\alpha^2 \left(\frac{n\pi}{a} \right)^2 x_1 - \beta^2 \left(\frac{m\pi}{b} \right)^2 x_2 \right].$$

It follows that the series (4) is uniformly convergent with respect to (s, t) and therefore it can be termwise integrated.

We obtain

$$u(x_1, x_2, z_1, z_2) = \int_0^a \int_0^b \left\{ \frac{4}{ab} \sum_{n,m=1}^{\infty} \exp \left[-\alpha^2 \left(\frac{n\pi}{a} \right)^2 x_1 - \beta^2 \left(\frac{m\pi}{b} \right)^2 x_2 \right] \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 \sin \frac{n\pi}{a} s \sin \frac{m\pi}{b} t \right\} \varphi(s, t) ds dt.$$

The Green's function is

$$(5) \quad G(x_1, x_2, z_1, z_2, s, t) = \frac{4}{ab} \sum_{n,m=1}^{\infty} \exp \left[-\alpha^2 \left(\frac{n\pi}{a} \right)^2 x_1 - \beta^2 \left(\frac{m\pi}{b} \right)^2 x_2 \right] \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 \sin \frac{n\pi}{a} s \sin \frac{m\pi}{b} t$$

and the solution of the problem is given by

$$(6) \quad u(x_1, x_2, z_1, z_2) = \int_0^a \int_0^b G(x_1, x_2, z_1, z_2, s, t) \varphi(s, t) ds dt.$$

In the case of a nonhomogeneous system

$$(7) \quad \begin{cases} u_{x_1} - \alpha^2 u_{z_1 z_1} = f_1(x_1, x_2, z_1, z_2), \\ u_{x_2} - \beta^2 u_{z_2 z_2} = f_2(x_1, x_2, z_1, z_2), \end{cases}$$

with the condition (2) and

$$(8) \quad u(0, 0, z_1, z_2) = 0, \quad (0 \leq z_1 \leq a, \quad 0 \leq z_2 \leq b),$$

the solution can be expressed as follows

$$(9) \quad \begin{aligned} u(x_1, x_2, z_1, z_2) = & \int_0^{z_1} \int_0^a \int_0^b \left\{ \frac{4}{ab} \sum_{n,m=1}^{\infty} \exp \left[-\beta^2 \left(\frac{m\pi}{b} \right)^2 (x_2 - t) \right] \times \right. \\ & \times \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 \sin \frac{n\pi}{a} s \sin \frac{m\pi}{b} t \left. \right\} f_2(x_1, t, p, q) dt dp dq + \\ & + \int_0^{z_2} \int_0^a \int_0^b \left\{ \frac{4}{ab} \sum_{n,m=1}^{\infty} \exp \left[-\alpha^2 \left(\frac{n\pi}{a} \right)^2 (x_1 - s) - \beta^2 \left(\frac{m\pi}{b} \right)^2 x_2 \right] \times \right. \\ & \times \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 \sin \frac{n\pi}{a} p \sin \frac{m\pi}{b} q \left. \right\} f_1(s, 0, p, q) ds dp dq. \end{aligned}$$

Using the Green's function (5), this solution takes the form

$$\begin{aligned}
 (10) \quad u(x_1, x_2, z_1, z_2) = & \int_0^{x_1} \int_0^a \int_0^b G(x_1-s, x_2, z_1, z_2, p, q) f_1(s, 0, p, q) ds dp dq + \\
 & + \int_0^{x_1} \int_0^a \int_0^b G(0, x_2-t, z_1, z_2, p, q) f_2(x_1, t, p, q) dt dp dq = \\
 & = \int_0^{x_1} \int_0^a \int_0^b \left[\int_0^{z_1} G(x_1-s, x_2, z_1, z_2, p, q) f_1(s, 0, p, q) ds + \right. \\
 & \left. + \int_0^{z_2} G(0, x_2-t, z_1, z_2, p, q) f_2(x_1, t, p, q) dt \right] dp dq.
 \end{aligned}$$

Remark. The uniqueness of the solution of problem (7) results by applying in an extended form the Principle of Maximum [2]. Indeed, if u_1 and u_2 are continuous solutions for $0 \leq x_1 \leq T$, $0 \leq x_2 \leq T$, $0 \leq z_1 \leq a$, $0 \leq z_2 \leq b$, of system (7) with the conditions (3) and

$$(11) \quad \begin{cases} u(x_1, x_2, 0, z_2) = g_1(x_1, x_2, z_2), \\ u(x_1, x_2, a, z_2) = g_2(x_1, x_2, z_2), \\ u(x_1, x_2, z_1, 0) = g_3(x_1, x_2, z_1), \\ u(x_1, x_2, z_1, b) = g_4(x_1, x_2, z_1), \end{cases}$$

then the function $v(x_1, x_2, z_1, z_2) = u_1 - u_2$ is solution of the problem

$$(12) \quad \begin{cases} v_{x_1} - \alpha^2 v_{z_1 z_1} = 0, \\ v_{x_2} - \beta^2 v_{z_2 z_2} = 0; \end{cases}$$

$$(13) \quad \begin{cases} v(x_1, x_2, 0, z_2) = 0, \\ v(x_1, x_2, a, z_2) = 0, \\ v(x_1, x_2, z_1, 0) = 0, \\ v(x_1, x_2, z_1, b) = 0 \end{cases} \quad \begin{cases} 0 < x_1 \leq T, & 0 < x_2 \leq T, \\ 0 < z_1 < a, & 0 < z_2 < b; \end{cases}$$

$$(14) \quad v(0, 0, z_1, z_2) = 0, \quad (0 \leq z_1 \leq a, \quad 0 \leq z_2 \leq b).$$

Applying the Principle of Maximum we get

$$v(x_1, x_2, z_1, z_2) \equiv 0$$

and therefore

$$u_1 \equiv u_2 \quad \text{for} \quad 0 \leq x_1 \leq T, \quad 0 \leq x_2 \leq T, \quad 0 \leq z_1 \leq a, \quad 0 \leq z_2 \leq b.$$

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FUNCTIA GREEN PENTRU UN SISTEM BI-PARAMETRIC DE EVOLUȚIE

(Rezumat)

Se indică funcția Green pentru problema Cauchy atașată sistemelor de ecuații de evoluție cu doi parametri și se obține forma concretă a unui semigrup bi-parametric de operatori.

SOME ASPECTS OF THE BOUNDEDNESS OF SOLUTIONS FOR ITO EQUATIONS

BY

MARICA LEWIN

Let $\{\Omega, K, P\}$ be a probability field. Let us consider the equation

$$(1) \quad \begin{cases} d(x(t, \omega)) = f(t, x) dt + \sigma(t, x) dw(t), \\ x(t_0) = x_0, \quad x_0 \in R^n \end{cases}$$

where $w(t, \omega)$ is a standard, Wiener n -dimensional process, $f(t, x)$ is a measurable function in (t, x) from $R^+ \times R^n$ in R^n , $\sigma(t, x)$ an $n \times n$ matrix whose components are measurable functions in (t, x) . We associate a family of nonanticipative σ -algebras F_t , $t \in [0, \infty)$ with the process $w(t, \omega)$. For instance F_t can be the least σ -algebra generated by $w(s)$, $0 \leq s \leq t$.

We assume that all the conditions from the theorem of existence and uniqueness of the solution are satisfied, although one can assume only the existence of the solution.

Let $V(t, x)$ be a function on $R^+ \times R^n \rightarrow R^+$, which belongs to C^1 in t and t_0, C^2 in x . We call such a function a Liapunov function.

The Liapunov function $V(t, x)$ satisfies Itô's formula

$$(2) \quad V(t, x(t, \omega)) - V(s, x(s, \omega)) = \int_s^t UV(u, x(u)) du + \int_s^t \left(\frac{\partial V}{\partial x}, u, x(u) \right) dw,$$

where

$$(3) \quad UV(t, x) = \frac{\partial V(t, x)}{\partial t} + \left(\frac{\partial V(t, x)}{\partial x}, f(t, x) \right) + \frac{1}{2} \sum_{i,j,m} \frac{\partial^2 V}{\partial x_i \partial x_j} \sigma_{im} \sigma_{jm}(t, x),$$

σ_{im} being the elements of the matrix $\sigma(t, x)$ and (x, y) the scalar product of the vectors x, y .

Moreover, we assume that

$$(4) \quad \left| \frac{\partial V}{\partial x} \right| \leq 1 + \|x\|, \quad x \in R^n,$$

condition which ensure some properties of the Itô integral.

Definitions. **Definition 1.** The solutions of Eq. (1) are ultimately bounded in probability for bound B , if there exists $B > 0$, such that $\lim_{t \rightarrow \infty} P_{x_0} \{ \|x(t, t_0, x_0)\| > B \} = 0$, B being the same for all solutions of the system. R_{x_0} is the probability which to the initial condition t_0 .

Definition 2. The solutions of the system (1) are equi-ultimately bounded in probability, if there is $B > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{x_0 \in S_\alpha} P \{ \|x(t, t_0, x_0)\| > B \} = 0, \quad \text{where } S_\alpha = \{x, \|x\| \leq \alpha\}.$$

Definition 3. The solutions of sistem (1) are ultimately bounded in probability if

$$\lim_{R \rightarrow \infty} \sup_{t_0 > 0} \lim_{t \rightarrow \infty} \sup_{x_0 \in S_\alpha} P \{ \|x(t, t_0, x_0)\| > R \} = 0.$$

We shall try to extend these definitions and to give more general conditions to ensure for the solutions of (1) the properties (1), (2), (3).

Definition 4. An ϵ -neighborhood of a set F relative to an open set Q is an open set $N_\epsilon(F)$, such that $N_\epsilon(F) = \{x \in Q \text{ such that } \|x - y\| < \epsilon \text{ for some } y \in F\}$.

Remark. One may consider ϵ -neighborhoods of a set F with respect to R^n , and then $F \subset R^n$.

Definition 5. The solutions of system (1) are ultimately bounded in probability with respect to F , if

$$\forall \epsilon > 0, \quad \lim_{t \rightarrow \infty} P_{x_0} \{ \omega, d(x(t, t_0, x_0), F) \geq \epsilon \} = 0,$$

where $d(x, F) = \inf_{y \in F} \|x - y\|$, the distance from x to F .

Definition 6. The solution $x(t, t_0, x_0)$ of the system (1) are equi-ultimately bounded with respect to F , if

$$\lim_{t \rightarrow \infty} \sup_{x_0 \in S_\alpha} P \{ \omega, d(x(t, t_0, x_0), F) \geq \epsilon \} = 0.$$

Definition 7. A scalar function $W(x)$ defined for $x \in Q$ is said to be positive definite with respect to a set S , if $W(x) = 0$ for $x \in S$ and if corresponding to each $\epsilon > 0$ and each compact set θ in Q , there exists a positive number $\delta(\epsilon, \theta)$ such that (see [2])

$$(5) \quad W(x) \geq \delta(\epsilon, Q^0) \quad \text{for } x \in \theta^0 - N(\epsilon, S).$$

Theorem 1. Let V be a Liapunov function. If

$$(a) \quad \lim_{t \rightarrow \infty} \inf_{\|x\| > r} V(t, x) = \infty;$$

(b) $UV \leq -W(x)$, where $W(x)$ is a positive definite function with respect to the bounded set $F \subset R^n$, then the solutions of Eq. (1) are ultimately bounded with respect to F .

The proof is a direct consequence of the following

Lemma 1 (see [4, p. 38]). Let $V(t, x)$ be nonnegative in an open region Q (Q is not necessarily bounded), x_t a right continuous strong Markov process

in Q and τ the random first exit time from Q . Let $V(t, x)$ be in the domain of A_0 , the weak infinitesimal operator of the process A_0 .

Let $A_0 V(t, x) \leq -b < 0$ in P . Then the average time $E_x \tau$ spent interior to P up to the first exit time is no greater than $V(x)/b$ (for the proof of the lemma, see [4, p. 38]).

P r o o f. Since the condition b) implies $UV \leq 0$ in R^n and from a) it results that the solutions of the system (1) are strongly bounded in probability, that is

$$(6) \quad \lim_{R \rightarrow \infty} \sup_{x_0 \in S_\alpha} P \left\{ \sup_{t > t_0} \|x(t, t_0, x_0)\| > R \right\} = 0.$$

So $\forall x_0 \in S_\alpha, \forall \varepsilon > 0, \exists R_0(\varepsilon, \alpha)$ so that $P \left\{ \sup_{t > t_0} \|x(t, t_0, x_0)\| \leq R_0 \right\} > 1 - \varepsilon$.

Since F is supposed bounded, we choose on $\varepsilon > 0$, so that $F \subset B_{R_0}(\varepsilon, \alpha)$, where $B_{R_0} = \{x, \|x\| \leq R_0\}$.

We consider a δ -neighborhood of F , $N(F, \delta)$ and let χ be the indicator function of the set $B_{R_0} - N(F, \delta)$,

$$\chi(s, \omega, \delta) = \begin{cases} 1 & \text{if } x(s, \omega) \in B_{R_0} - N(F, \delta), \\ 0 & \text{if } x(s, \omega) \notin B_{R_0} - N(F, \delta). \end{cases}$$

Because of the condition b) which implies Lemma 1 the time spent by the solution in $B_{R_0} - N(F, \delta)$ is finite.

Let

$$T_{x_0}(t, \delta, \omega) = \int_{t > \tau_{R_0}}^{\tau_{R_0}} \chi(s, \omega, \delta) ds$$

be the time spent by the solution with the initial condition x_0 , in the region $B_{R_0} - N(F, \delta)$ after time t and before either $t \rightarrow \infty$ or the first exit time from B_{R_0} .

For $t \rightarrow \infty$

$$(7) \quad T_{x_0}(t, \omega, \delta) \rightarrow 0 \text{ with probability } 1, \forall x_0 \in S_\alpha.$$

Let Δh be

$$\Delta h = \sup_{|t_1 - t_2| \leq h} \|x(t_1, t_0, x_0) - x(t_2, t_0, x_0)\|,$$

Δh approaches 0 for $h \rightarrow 0$, $\forall [t_1, t_2] \subset [t_0, T]$, $\forall T \in R^+$ a.e. This implies

$$(8) \quad \lim_{h \rightarrow 0} P \left(\Delta h > \frac{\delta}{2} \right) = 0, \forall \delta > 0.$$

From the triangular inequality it results

$$(9) \quad d(x(s, t_0, x_0), F) \geq d(x(t), F) - \sup_{|t-s| \leq h} \|x(t) - x(s)\|, \forall s \in [t, t+h],$$

where h is chosen so that $P \left(\Delta h > \frac{\delta}{2} \right) < \varepsilon_1$, ε_1 small enough.

We denote by

$$(10) \quad \begin{cases} A = \{\omega, d(x(t, F)) \geq \delta\}, \\ A_1 = \left\{ \omega, d(x(s, F)) \geq \frac{\delta}{2}, \forall s \in [t, t+h] \right\}, \\ A_2 = \left\{ \omega, \Delta h \leq \frac{\delta}{2} \right\}. \end{cases}$$

We obtain $A \cap A_2 \subseteq A_1$, which implies $P(A \cap A_2) \leq P(A_1)$, or because A and A_2 are independent events, $P(A)P(A_2) \leq P(A)$.

But $A_1 \subset B_1 \cup B_2$ where

$$B_1 = \left\{ \omega, x(s, \omega) \in B_{R_*} - N\left(\frac{\delta}{2}, F\right), s \in \cup [t_k, t_{k+1}] \subset [t, t+h] \right\}$$

an at most countable union

$$B_2 = \{\omega, \|x(s, \omega)\| > R_*, \text{ for } s \in [t, t+h] \cup [t_k, t_{k+1}]\};$$

because of the continuity with probability of the Itô process, the solution can not move infinitely often from $B_{R_*} - N\left(\frac{\delta}{2}, F\right)$ in CB_{R_*} .

But

$$B_1 \subset \left\{ \omega, T_{x_*}\left(t, \frac{\delta}{2}, \omega\right) > 0 \right\}$$

and

$$B_2 \subset \left\{ \omega, \sup_{t > t_0} \|x(t, t_0, x_0)\| > R_0 \right\}.$$

So

$$(11) \quad P(A)P(A_2) \leq P(A_1) \leq P\left\{ \omega, T_{x_*}\left(t, \frac{\delta}{2}, \omega\right) > 0 \right\} + P\left\{ \omega, \sup_{t > t_0} \|x(t, t_0, x_0)\| > R_0 \right\}.$$

From (7) and because $P(A_2) > 1 - \varepsilon$, $\forall \varepsilon_1$ small enough, we have

$$(12) \quad \lim_{t \rightarrow \infty} P\{\omega, d(x(t), F) > \delta\} \leq P\left\{ \omega, \sup_{t > t_0} \|x(t)\| > R_0 \right\}.$$

In (12) taking the limit for $R_0 \rightarrow \infty$, we obtain

$$\lim_{t \rightarrow \infty} P\{\omega, d(x(t, t_0, x_0), F) \geq \delta\} = 0, \forall \delta > 0 \text{ small enough,}$$

which is the conclusion of the Theorem.

Corollary. If $x_0 \in R^n \setminus F$, then we can consider $V(t, x) \leq 0$ on F . We define $\tau_F = \{\inf(t, x_t) \notin R^n - F\}$ and we consider the stopped process $x_t \cap \tau_F$. Then $UV(t \cap \tau_F, x(t \cap \tau_F))$ is meaningful and there is a random variable $\tau \leq \infty$ such that $x_t \rightarrow F$ in probability, as $t \rightarrow \tau$.

Remark. This theorem is a light generalisation of Kushner's theorem, because the function W is positive definite only on compacts, not on the whole region of definition of $V(t, x)$.

Definition. If M is a closed set in R^n such that $P_{x,s}\{x(t) \in M \text{ for some } t > s\} = 0$, $X \notin M$, M is *nonattainable* by $x(t, \omega)$.

In what follows, we assume conditions for the Liapunov function $V(t, x)$ only on $R^n - F$, where F is $\{x \in R^n \mid \|x\| > R_0\}$.

Theorem 2. Let $V(t, x)$ be a Liapunov function such that

a) $\liminf_{r \rightarrow \infty} V(t, x) = \infty$, $r > R_0$, R_0 a real number which can be large;

b) $UV(t, x) \leq -cV(t, x) + \gamma$, for $\|x\| \geq R_0$, and γ any real number.

Let $\alpha > R_0$. If $S_\alpha = \{x, \|x\| > \alpha\}$ is nonattainable by the solution $x(t, \omega)$, then the solution of the system (1) is α -ultimately bounded in probability.

Proof. Let $\alpha > R_0$, and $\tau_\alpha(\omega) = \{\sup_t \|x(t, t_0, x_0, \omega)\| < \alpha\}$ be. Consider

$$\tau_\alpha \wedge t = \min(t, \tau_\alpha(\omega, t_0, x_0)), \quad t \in R^+.$$

For a fixed, but arbitrarily chosen t , $\tau_\alpha(\omega) \wedge t$ is a Markov time with respect to the Markov process $x(t, \omega)$, the solution of the system (1).

We denote by $F_{\tau_\alpha \wedge t}$ the σ -algebra of all events $A \in K$, so that $A \cap (\tau_\alpha \wedge t \leq v)$ is in F_v , for all $v \geq t_0$, $\{F_v, v \geq t_0\}$ being the family of the nonanticipating σ -algebras defined at the beginning.

Let us apply Ito's formula to the function $V(t, x)$, $x(t, \omega)$ being the solution of the system (1) on the stochastic interval $[\tau_\alpha \wedge t + s, \tau_\alpha \wedge t + s + h]$; $s, h \in R^+$:

$$(13) \quad \begin{aligned} & V(\tau_\alpha \wedge t + s + h, x(\tau_\alpha \wedge t + s + h)) - V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s, t_0, x_0)) = \\ & \int_{\tau_\alpha \wedge t + s}^{\tau_\alpha \wedge t + s + h} UV(u, x(u, \omega)) du + \int_{\tau_\alpha \wedge t + s}^{\tau_\alpha \wedge t + s + h} \frac{\partial V}{\partial x}(u, x(u, \omega)) b(u, x(u, \omega)) d\omega(u). \end{aligned}$$

Let $\chi_{\tau_\alpha \leq t}$ be the characteristic function of the set $\{\tau_\alpha(\omega, t_0, x_0) \leq t\}$. Multiplying the relationship (13) by $\chi_{\tau_\alpha \leq t}$ and taking the conditional mean with respect to the σ -algebra $F_{\tau_\alpha \wedge t}$, we obtain

$$(14) \quad \begin{aligned} & E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s + h; x(\tau_\alpha \wedge t + s + h)) / F_{\tau_\alpha \wedge t}] = \\ & = E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s)) / F_{\tau_\alpha \wedge t}] = \\ & = E \left[\chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t + s}^{\tau_\alpha \wedge t + s + h} UV(u, x(u, \omega)) du / F_{\tau_\alpha \wedge t} + \right. \\ & \left. + E \left[\chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t + s}^{\tau_\alpha \wedge t + s + h} \frac{\partial V}{\partial x}(u, x(u, \omega)) b(u, x(u, \omega)) d\omega / F_{\tau_\alpha \wedge t} \right] \right]. \end{aligned}$$

The function $\chi_{\tau_\alpha \leq t}$ is $F_{\tau_\alpha \wedge t}$ measurable. Also, for two Markov times, so that $P(\sigma \geq v) = 1$, $F_v \leq F_\sigma$. This implies $F_{\tau_\alpha \wedge t} \leq F_{\tau_\alpha \wedge t + s}$, $\forall s \geq 0$. Using also

the property of the conditional mean $E[x/F_2] = E\{E[x/F_1]/F_2\}$ for the σ -fields $F_2 \leq F_1$ and the fact that we are dealing with an Itô integral, we have

$$(15) \quad E \left[\chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} \frac{\partial V}{\partial x} b(u, x(u, \omega)) dw(v) / F_{\tau_\alpha \wedge t} \right] = \\ = \chi_{\tau_\alpha \leq t} \left\{ E \left[E \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} \frac{\partial V}{\partial x} b(u, x(u, \omega)) dw(u) / F_{\tau_\alpha \wedge t+s} \right] / F_{\tau_\alpha \wedge t} \right\} = 0;$$

(14) becomes

$$(16) \quad E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t+s+h, x(\tau_\alpha \wedge t+s+h) / F_{\tau_\alpha \wedge t}] - E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t+s, x(\tau_\alpha \wedge t+s) / F_{\tau_\alpha \wedge t}] = \\ = E \left[\chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} UV(u, x(u, \omega)) du / F_{\tau_\alpha \wedge t} \right].$$

Dividing (16) by h and taking the limit for $h \rightarrow 0$ we have

$$(17) \quad \lim_{h \rightarrow 0} \frac{1}{h} \{ E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t+s+h, x(\tau_\alpha \wedge t+s+h) / F_{\tau_\alpha \wedge t}] - \\ - E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t+s, x(\tau_\alpha \wedge t+s) / F_{\tau_\alpha \wedge t})] \} = \\ = \lim_{h \rightarrow 0} \frac{1}{h} \left\{ E \left[\chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} UV(u, x(u, \omega)) du / F_{\tau_\alpha \wedge t} \right] \right\}.$$

Applying the condition c) of the theorem and Lebesgue's convergence theorem for the conditional mean, we obtain

$$\lim_{h \rightarrow 0} \frac{1}{h} \left\{ E \left[\chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} UV(u, x(u, \omega)) du / F_{\tau_\alpha \wedge t} \right] \right\} \leq \\ \leq -c \lim_{h \rightarrow 0} \left\{ \frac{1}{h} E \chi_{\tau_\alpha \leq t} \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} V(u, x(u, \omega)) du / F_{\tau_\alpha \wedge t} \right\} = \\ = -c \chi_{\tau_\alpha \leq t} E \left[\lim_{h \rightarrow 0} \frac{1}{h} \int_{\tau_\alpha \wedge t+s}^{\tau_\alpha \wedge t+s+h} V(u, x(u, \omega)) du / F_{\tau_\alpha \wedge t} \right].$$

This leads to

$$(18) \quad \lim_{h \rightarrow 0} \frac{1}{h} \{ E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s + h, x(\tau_\alpha \wedge t + s + h)) / F_{\tau_\alpha \wedge t}] - \\ - E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s)) / F_{\tau_\alpha \wedge t}] \leq - \\ - c E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s)) / F_{\tau_\alpha \wedge t}] + \gamma, \quad \forall s \geq 0,$$

with probability 1.

We denote by

$$H(s) = E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s)) / F_{\tau_\alpha \wedge t}] + \\ + c \int_0^s E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + u, x(\tau_\alpha \wedge t + u)) / F_{\tau_\alpha \wedge t}] du - \gamma s.$$

The continuity of the solution of the Itô equation and the properties of the Liapunov function V , implies the continuity of the function $H(s, \omega)$ with probability 1.

The continuity of $H(s, \omega)$ and the relationship (18) implies that $H(s)$ is a monotonously decreasing function $\forall s \geq 0$, with probability 1.

So $H(s, \omega) \leq H(0, \omega)$, $\forall s \geq 0$. Hence

$$(19) \quad E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s)) / F_{\tau_\alpha \wedge t}] \leq \chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t, x(\tau_\alpha \wedge t)) - \\ - c \int_0^s E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + u, x(\tau_\alpha \wedge t + u)) / F_{\tau_\alpha \wedge t}] du + \gamma s,$$

because $H(0) = E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t, x(\tau_\alpha \wedge t, t_0, x_0, \omega))]$ and $\chi_{\tau_\alpha \wedge t} V(\tau_\alpha \wedge t, x(\tau_\alpha \wedge t))$ is $F_{\tau_\alpha \wedge t}$ measurable.

From (19), applying Gronwall's lemma, we obtain

$$(20) \quad E[\chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t + s, x(\tau_\alpha \wedge t + s)) / F_{\tau_\alpha \wedge t}] \leq \\ \leq \chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t, x(\tau_\alpha \wedge t)) e^{-cs} + \frac{\gamma}{c}, \quad \forall s \geq 0;$$

(20) takes place with probability 1.

Hence, (20) is true also for $s = t - t \wedge \tau_\alpha(\omega)$.

For s chosen above, (20) becomes

$$(21) \quad E[\chi_{\tau_\alpha \leq t} V(t, x(t, t_0, x_0, \omega)) / F_{\tau_\alpha \wedge t}] \leq \chi_{\tau_\alpha \leq t} V(\tau_\alpha \wedge t, x(\tau_\alpha \wedge t)) e^{-c(t-t \wedge \tau_\alpha)}$$

with probability 1.

On the set $\{\tau_\alpha(\omega) \leq t\}$, $\|X(\tau_\alpha \wedge t)\| = \|X(\tau_\alpha(\omega))\| = \alpha$,

it results that there is a real function $b(r, t)$ positive and continuous, so that $V(t, x(t)) \leq b(\|x\|, t)$. So $V(\tau_\alpha \wedge t, x(\tau_\alpha \wedge t)) \leq b(\|x(\tau_\alpha)\|) = b(\alpha)$ on the set $(\tau_\alpha \leq t)$ and we can write

$$(22) \quad E[\chi_{\tau_\alpha \leq t} V(t, x(t, t_0, x_0, \omega)) / F_{\tau_\alpha \wedge t}] \leq b(\alpha) \chi_{\tau_\alpha \leq t} e^{-c(t - \tau_\alpha \wedge t)}.$$

Since $E\{E[y/F]\} = Ey$, $\forall \sigma$ -field $F \subset K$, taking the mean in (22), we get

$$(23) \quad E\chi_{\tau_\alpha \leq t} V(t, x(t, t_0, x_0, \omega)) \leq b(\alpha) E\chi_{\tau_\alpha \leq t} e^{-c(t - \tau_\alpha \wedge t)} + \frac{\gamma}{c}.$$

The condition a) of the theorem implies that there is a continuous, positive function $a(r)$ monotoniously increasing, so that

$$(24) \quad \lim_{n \rightarrow \infty} a(r) = \infty,$$

$$(25) \quad a(\|x\|) \leq V(t, x(t, t_0, x_0, \omega));$$

(24) ensures that there is a $\beta_0 > 0$, so that $a(\beta) > 0$, for $\beta > \beta_0$.

Choosing $\beta > \max(\beta_0, \alpha)$ from (24) and (25) we have

$$a(\beta) P(\|x(t, t_0, x_0, \omega)\| > \beta, \tau_\alpha \leq t) \leq b(\alpha) E\chi_{\tau_\alpha \leq t} e^{-c(t - \tau_\alpha \wedge t)}$$

from which it results

$$(26) \quad \begin{aligned} & \sup_{x_0 \in S_\alpha} P(\|x(t, t_0, x_0, \omega)\| > \beta, \tau_\alpha(t_0, x_0, \omega) \leq t) \leq \\ & \leq \frac{b(\alpha)}{a(\beta)} \sup_{x_0 \in S_\alpha} E\chi_{\tau_\alpha \leq t} e^{-c(t - t \wedge \tau_\alpha)} + \frac{\gamma}{c}. \end{aligned}$$

Because Lebesgue's theorem of bounded convergence ensures

$$\lim_{t \rightarrow \infty} E\chi_{\tau_\alpha \leq t} e^{-c(t - t \wedge \tau_\alpha)} = 0; \quad \forall x_0 \in S_\alpha,$$

we get from (26)

$$\lim_{t \rightarrow \infty} \sup_{x_0 \in S_\alpha} P(\|x(t, t_0, x_0, \omega)\| > \beta, \tau_\alpha(\omega) \leq t) \leq \frac{\gamma}{ca(\beta)}$$

hence

$$(27) \quad \lim_{\beta \rightarrow \infty} \sup_{x_0 \in S_\alpha} \lim_{t \rightarrow \infty} \sup_{x_0 \in S_\alpha} P(\|x(t, t_0, x_0, \omega)\| > \beta, \tau_\alpha(\omega) \leq t) = 0$$

Thus

$$\begin{aligned} P(\|x(t, t_0, x_0, \omega)\| > \beta) &= P(\|x(t, t_0, x_0, \omega)\| > \beta, \tau_\alpha \leq t) + \\ &+ P(\|x(t, t_0, x_0, \omega)\| > \beta, t < \tau_\alpha(\omega)). \end{aligned}$$

Since

$$\lim_{\beta \rightarrow \infty} \sup_{t_0} \lim_{t \rightarrow \infty} \sup_{x_0 \in S_\alpha} P(\|x(t, t_0, x_0, \omega)\| > \beta, \tau_\alpha > t) = 0,$$

(27) implies the conclusion of the theorem.

In some cases only properties of boundedness of some components of the solution are to be considered.

For instance, let us take the system

$$(28) \quad \begin{aligned} dx(t) &= f_1(t, x, y)dt + \sigma_1(t, x, y)dW_1, \\ dy(t) &= f_2(t, x, y)dt + \sigma_2(t, x, y)dW, \end{aligned}$$

where $x \in R^n$, $y \in R^m$, $W_1(t, \omega)$, $W_2(t, \omega)$ are n and m dimensional Brownian motions, and $f_1, f_2, \sigma_1, \sigma_2$ have all the known properties.

We are interested only in ultimate boundedness of the component x to a set F , bounded. In fact, the case where some components of the Markov process are of no interest is probably the general case. Some of the states of a Markov model of a control process will be the observed states or the quantities of interest in the control process. However in order to „Markovianize“ a control process, many „states“ may have to be added.

Hence the solution of the system (28) is α -ultimately bounded with respect to the component x , if

$$\lim_{R \rightarrow \infty} \sup_{t_0} \lim_{t \rightarrow \infty} \sup_{x_0, y_0 \in S_\alpha} P\{\|x(t, t_0, x_0, \omega)\| > R\} = 0.$$

We can prove

Theorem 3. Let $V(t, x, y)$ be a Liapunov function defined on $R^+ \times R^n \times R^m \rightarrow R$. If

- $\liminf_{r \rightarrow \infty} V(t, x, y) = \infty$, $r > R_0$, R_0 a real number which can be large;
- $UV(t, x, y) \leq -cV(t, x, y) + \gamma$, for $\|x\|^2 + \|y\|^2 \geq R_0^2$, and γ any real number.

If for $\alpha > R_0$ $S_\alpha = \{x, \|x\| > \alpha\}$ is nonattainable by $\{x(t), y(t)\}$ then the solution of the system (28) is α -ultimately bounded with respect to the component x .

Likewise, we define the α -ultimately boundedness with respect to the component y , or with respect to the both components.

Remark 1. In the proof of the Theorem 2 it is possible to invert the conditional mean with the Lebesgue's integral and to apply directly Gronwall's lemma.

Remark 2. Because we supposed the set S_α nonattainable, $\alpha(\omega) = \inf\{t, \|x(t), t_0, x_0, \omega\| \geq \alpha\}$, and then $\tau_\alpha(\omega)$ is a Markov time. Usually, $\alpha(\omega) = \sup\{t, \|x(t, t_0, x_0, \omega)\| < \alpha\}$ is not a Markov time.

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UNELE ASPECTE PRIVIND MĂRGINIREA SOLUȚIILOR ECUAȚIEI LUI ITÔ

Rezumat

În lucrarea de față sint prezentate proprietăți de mărginire la limită pentru soluțiile ecuației Itô. Aceste proprietăți sint studiate cu ajutorul unei funcții Liapunov care satisface anumite condiții doar în exteriorul unor mulțimi mărginite și închise.

STABILIREA NUMĂRULUI OPTIM DE PUNCTE CUNOSCUTE DIN CARE SE FACE DETERMINAREA VECTORULUI DEPLASĂRII UNEI CONSTRUCȚII PRIN INTERSECȚIE MULTIPLĂ ÎNAINTE

DE

GH. NISTOR

La determinarea componentelor orizontale ale vectorului deplasării unei construcții prin metoda intersecției multiple înainte, tratată prin metoda riguroasă a celor mai mici pătrate, trebuie avut în vedere necesitatea unui surplus optim de date. Pentru aceasta, se va stabili o relație directă între precizia de determinare a vectorului deplasării orizontale a unui punct de control, de pe construcția luată în studiu, și numărul de puncte cunoscute din care se face determinarea.

În cazul determinării componentelor vectorului deplasării unui punct de control dintr-un număr de $n+1$ puncte cunoscute, din rețeaua de micro-triangulație, se scriu relațiile de deformații [1] exprimate de ecuația matriceală

$$(1) \quad A_{n+1,2} X_{2,1} + L_{n+1,1} = V_{n+1,1}, \quad \text{cu ponderea } P_{n+1,n+1},$$

unde

$$A_{n+1,2} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_{n+1} & b_{n+1} \end{bmatrix}, \quad X_{2,1} = \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix}, \quad L_{n+1,1} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \\ l_{n+1} \end{bmatrix},$$

$$V_{n+1,1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ v_{n+1} \end{bmatrix}, \quad P_{n+1,n+1} = \begin{bmatrix} p_1 & 0 & \dots & 0 & 0 \\ 0 & p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p_n & 0 \\ 0 & 0 & \dots & 0 & p_{n+1} \end{bmatrix}.$$

Deoarece deplasările orizontale au valori în general mici, de ordinul milimetrilor sau al zecilor de milimetri, iar distanțele de la punctele cunoscute la punctele de control sînt relativ mici, coeficienții de direcție se calculează în milimetri cu ajutorul relațiilor

$$(2) \quad a \left[\frac{\text{cc}}{\text{mm}} \right] = - \rho^{\text{oc}} \frac{\sin \theta_i}{D_i (\text{mm})} = -0,63662 \frac{\sin \theta_i}{D_i (\text{km})},$$

$$(3) \quad b \left[\frac{\text{cc}}{\text{mm}} \right] = \rho^{\text{cc}} \frac{\cos \theta_i}{D_i (\text{mm})} = -0,63662 \frac{\cos \theta_i}{D_i (\text{km})}.$$

În acest fel, în relațiile de deformare toate mărimile liniare sînt exprimate în milimetri și toate mărimile unghiulare în secunde centesimale. Termenii liberi și corecțiile sînt exprimate prin relațiile

$$(4) \quad l_i = l_{1,i} - l_{0,i} = \theta_i^1 - \theta_i^0, \quad i = 1, 2, \dots, n+1,$$

$$(5) \quad v_i = v_{1,i} - v_{0,i}.$$

Determinarea componentelor vectorului deplasării orizontale a punctului de control, produsă între două cicluri de observații, se face sub condiția

$$(6) \quad V_{1, n+1}^T P_{n+1, n+1} V_{n+1, 1} \rightarrow \min.$$

Condiția de minim se va obține prin anularea derivatelor parțiale, ajungînd la sistemul ecuațiilor normale ale componentelor deplasării, exprimat de ecuația matriceală

$$(7) \quad N_{2,2} X_{2,1} + A_{2,n+1}^T P_{n+1, n+1} L_{n+1,1} = 0,$$

unde

$$N_{2,2} = \begin{bmatrix} [aap] & [abp] \\ [abp] & [bbp] \end{bmatrix}, \quad X_{2,1} = \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix}, \quad A_{2,n+1}^T P_{n+1, n+1} L_{n+1,1} = \begin{bmatrix} [alp] \\ [blp] \end{bmatrix}.$$

Evaluarea preciziei componentelor deplasării se face cu ajutorul matricei [2]

$$(8) \quad M_x^2 = m_0^2 Q_{2,2},$$

unde m_0 reprezintă eroarea unității de pondere iar $Q_{2,2} = N_{2,2}^{-1}$ este matricea coeficienților de pondere.

Erorile medii pătratice ale componentelor vectorului deplasării, de-a lungul axelor de coordonate, sînt exprimate de formulele

$$(9) \quad m_{\Delta x} = \pm m_0 \sqrt{Q_{11}} = \pm m_0 \sqrt{\frac{1}{p_{\Delta x}}},$$

$$(10) \quad m_{\Delta y} = \pm m_0 \sqrt{Q_{22}} = \pm m_0 \sqrt{\frac{1}{p_{\Delta y}}},$$

iar eroarea medie pătratică a vectorului deplasării orizontale este dată de

$$(11) \quad m_L = \sqrt{m_{\Delta x}^2 + m_{\Delta y}^2} = m_0 \sqrt{Q_{11} + Q_{22}} = m_0 \sqrt{\frac{1}{p_{\Delta x}} + \frac{1}{p_{\Delta y}}}.$$

Exprimînd inversele ponderilor în funcție de orientările dreptelor de determinare și apoi în funcție de unghiurile γ_j , ($j=1, 2, \dots, n$), sub care se intersectează dreptele de determinare, precum și în funcție de distanțele D_i ($i=1, 2, \dots, n+1$), de la punctele cunoscute ale rețelei de microtriangulație pînă la punctul de control și admitînd în final că $\gamma_1 \approx \gamma_2 \approx \dots \approx \gamma_n \approx \gamma$ și $D_1 \approx D_2 \approx \dots \approx D_{n+1} \approx D$, cu relația (11) se obține pentru evaluarea preciziei

de determinare a vectorului deplasării orizontale a punctului de control prin intersecție multiplă înainte, expresia

$$(12) \quad m_L = \frac{m_0}{\rho^{\circ 0}} D \left[\frac{n}{(n-1)\sin^2\gamma + (n-2)\sin^2 2\gamma + \dots + \sin^2(n-1)\gamma} \right]^{\frac{1}{2}},$$

în care n reprezintă numărul dreptelor de determinare.

S-a întocmit tabelul 1 cuprinzînd erorile medii pătratice ale vectorului deplasării orizontale a punctului de control pentru diferite numere de puncte vechi (sau drepte de determinare) din care se face urmărirea comportării construcției. Calculul erorilor medii pătratice s-a făcut luînd $m_0 = \pm 3^{\circ}$, $D = 100$ m și $\gamma = 100^{\circ}$.

Tabelul 1

Nr. puncte vechi n	2	3	4	5	6	7	8	9	10
Eroare medie pătratică m_L (mm)	0,66	0,58	0,47	0,43	0,39	0,36	0,33	0,32	0,30

Datele tabelului au fost reprezentate grafic (fig. 1).

Analiza datelor tabelului și a graficului arată că nu este recomandabilă creșterea nelimitată a numărului de puncte vechi din care se face urmărirea comportării construcției deoarece, la un număr de puncte vechi $n > 4$, creșterea preciziei devine foarte mică, ne semnificativă și deci neeconomică.

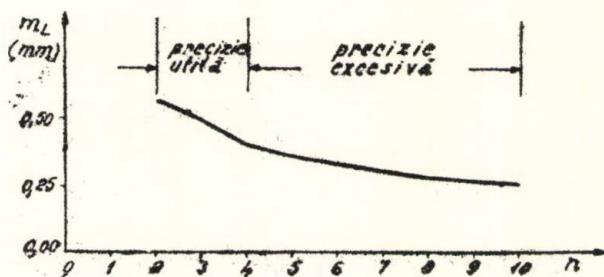


Fig. 1

Rezultă că la determinarea vectorului deplasării orizontale a punctului de control de pe construcția observată, se va avea în vedere obținerea unei precizii utile, obținută prin folosirea a trei, maximum patru puncte cunoscute, mărirea preciziei obținîndu-se prin micșorarea erorii unității de pondere m_0 , respectiv prin mărirea preciziei de măsurare pe teren a unghiurilor orizontale β_i , în ciclurile de observații inițial și actual.

Concluzii. 1. Studiul efectuat dă posibilitatea stabilirii numărului optim de puncte cunoscute ale rețelei de microtriangulație, din care se efectuează urmărirea comportării unei construcții. Această are ca efect importante

economii atât în ce privește volumul lucrărilor de teren cât și al lucrărilor de birou.

2. Cercetările au arătat că obținerea unei precizii corespunzătoare se obține nu prin mărirea excesivă a surplusului de date, reprezentat de un număr mare de puncte cunoscute, ci prin mărirea preciziei de măsurare a unghiurilor orizontale în toate ciclurile de observații.

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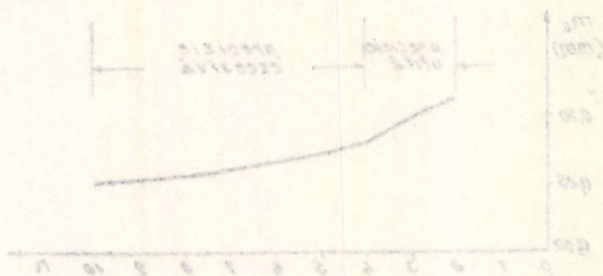
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ÉTABLISSEMENT DU NOMBRE OPTIMAL DE POINTS CONNUS POUR DÉTERMINER LE VECTEUR DÉPLACEMENT D'UNE CONSTRUCTION PAR INTERSECTION MULTIPLE EN AVANCE

(Résumé)

Pour déterminer le vecteur déplacement d'une construction à l'aide de la méthode rigoureuse des moindres carrés on doit tenir compte de la nécessité de disposer d'un nombre optimal de données supplémentaires. Dans cette Note on établit une relation directe reliant la précision de la détermination du vecteur déplacement horizontal d'un point de contrôle de la construction, et le nombre des points connus d'où on réalise la détermination.



Rezultă că în determinarea vectorului deplasării orizontale a punctului de control de pe construcția observată se va avea în vedere obținerea unei precizii minime, obținută prin folosirea nr. maxim de puncte cunoscute, măsurate cu precizie, obținându-se prin măsurarea corectă unghiurilor de pondere, respectiv prin măsurarea preciziei de măsurare pe teren a unghiurilor orizontale și în celelalte de observații inițiale și actuale.

Concluzii. Studiul efectuat de posibilitățile stabilirii numărului optim de puncte cunoscute ale rețelei de măsurare, din care se determină vectorul deplasării orizontale a punctului de control, arată că este important

A NONLOCAL CONTINUUM WITH ASYMMETRIC STRESS TENSOR

BY

C. I. BORSȚ

The theory of nonlocal continuum had a very great development. Many papers were published in this field accompanied as well as by many points of view. Among these papers we quote [1]...[5].

In the paper [6] a new attempt to derive the equations of motion for an elastic nonlocal continuum with symmetric stress tensor was given.

In our description, for such media the factors which concur to deformation are divided into external and internal factors. Both, the external and internal factors can be represented by body forces per unit mass, body couples per unit mass, surface tractions per unit area, surface couples per unit area, heat sources per unit mass, magnetic field, etc.

Thus, we have internal factors as internal body forces, internal couples, internal surface couples, internal heat sources, etc.

In the mentioned paper [6] the body was subjected to external surface tractions and external body forces and the internal nonlocal reply of the body was represented by an internal body forces field.

In this paper we consider the case when the external factors received by the body are represented by external body forces, external body couples and external surface tractions. The internal reply of the body is represented by an internal body forces field and an internal body couples field.

Let V be the domain occupied by a continuous body and let $P \subset V$ be an arbitrary part of V and ∂P the boundary of P .

We suppose that the influence of the mass contained in the domain $V - P$ to the mass contained in the domain P can be described by an internal body forces field and an internal body couples field acting in the points of P . On the boundary ∂P we consider only the surface tractions.

We will employ the usual notations sometimes without explanation.

The continuum model which we intend to develop obeys to the following principles:

I. Principle of conservation of mass

$$(1) \quad \frac{d}{dt} \int_P \rho(x) dX = 0,$$

where $\rho(x)$ is the density of the medium. Here and in the following dX and dY are elements of volume.

II. Principle of internal body actions.

a) There exists a vectorial function $\mathbf{g}(x, y)$ defined on $V \times V$, called internal body forces density, such that, the internal body force $\mathbf{g}_p(x)$ in the point $x \in V$ caused by the mass contained in a region $P \subset V$ is given by

$$\rho(x)\mathbf{g}_p(x) = \int_P \rho(y)\mathbf{g}(x, y)dY \quad x \in V, \quad y \in P.$$

Thus

$$\rho(x)\mathbf{g}(x) = \rho(x)\mathbf{g}_v(x) = \int_V \rho(y)\mathbf{g}(x, y)dY$$

is the total internal body forces density per unit volume in the point $x \in V$.

b) There exists a vectorial function $\mathbf{m}(x, y)$ defined on $V \times V$, called internal body couples density, such that, the internal body couple $\mathbf{m}_p(x)$ in the point $x \in V$ caused by the mass contained in P is given by

$$\rho(x)\mathbf{m}_p(x) = \int_P \rho(y)\mathbf{m}(x, y)dY$$

and

$$\rho(x)\mathbf{m}(x) = \rho(x)\mathbf{m}_v(x) = \int_V \rho(y)\mathbf{m}(x, y)dY$$

is the total internal body couple in the point $x \in V$.

c) The functions $\mathbf{g}(x, y)$ and $\mathbf{m}(x, y)$ have the following properties :

$$(2) \quad \int_P \int_P \rho(y)\mathbf{g}(x, y)dX dY = \int_P \rho(x)\mathbf{g}_p(x) dX = 0,$$

$$(3) \quad \int_P \left[\mathbf{r} \times \int_P \rho(y)\mathbf{g}(x, y)dY \right] dX + \int_P \int_P \rho(y)\mathbf{m}(x, y)dX dY = 0,$$

where $\mathbf{r}$ is the position of the point x with respect to the coordinate origin.

The mechanical meaning of the conditions (2), (3) is evident.

III. Principle of linearmomentum. The derivative of the linear momentum of P with respect to the time is equal to the resultant of external (to P) forces

$$(4) \quad \frac{d}{dt} \int_P \rho(x)\mathbf{v}(x)dX = \int_P \rho(x)\mathbf{f}(x)dX + \int_P dX \int_{V-P} \rho(y)\mathbf{g}(x, y)dY + \int_{\partial P} \mathbf{t} d\sigma,$$

where $\mathbf{f}(x)$ is the external body force per unit mass, $\mathbf{t}$ the surface traction on ∂P per unit area and $\mathbf{v}$ the velocity.

IV. Principle of moment of momentum. The derivative of the moment of momentum of P with respect to the time is equal to the resultant moment of external (to P) applied actions. This means

$$(5) \quad \frac{d}{dt} \int_P [\mathbf{r} \times \rho(x) \mathbf{v}(x)] dX = \int_P [\mathbf{r} \times \rho(x) \mathbf{f}(x) + \rho(x) \mathbf{M}(x)] dX + \\ + \int_P \left[\mathbf{r} \times \int_{V-P} \rho(y) \mathbf{g}(x, y) dY \right] dX + \int_P dX \int_{V-P} \rho(y) \mathbf{m}(x, y) dY + \int_{\delta P} (\mathbf{r} \times \mathbf{t}) d\sigma,$$

where $\mathbf{M}(x)$ is the external body couple per unit mass.

Making use by these principles we can derive the equations of motion for our nonlocal continuum.

In account of (2) and (3) we have

$$(6) \quad \int_P dX \int_{V-P} \rho(y) \mathbf{g}(x, y) dY = \int_P \rho(x) \mathbf{g}(x) dX, \\ (7) \quad \int_P \left[\mathbf{r} \times \int_{V-P} \rho(y) \mathbf{g}(x, y) dY \right] dX + \int_P dX \int_{V-P} \rho(y) \mathbf{m}(x, y) dY = \\ = \int_P \rho(x) [\mathbf{r} \times \mathbf{g}(x) + \mathbf{m}(x)] dX$$

and with the equation of continuity

$$(8) \quad \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0,$$

derived from the first principle, the principle of linear momentum gives

$$(9) \quad \int_P [\mathbf{t}_{i,i} + \rho(\mathbf{f} + \mathbf{g}) - \rho \mathbf{a}] dX = 0,$$

where $\mathbf{a}$ is the acceleration field.

Supposing in (8) the integrand continuous, we have

$$(10) \quad \mathbf{t}_{i,i} + \rho(\mathbf{f} + \mathbf{g}) = \rho \mathbf{a},$$

because (9) must be valid for arbitrary P .

In a rectangular Cartesian sistem of coordinate, we obtain the well-known equations of motion

$$\sigma_{ji,j} + \rho(f_i + g_i) = \rho a_i,$$

where σ_{ij} are the components of the stress tensor.

With the help of (8), (10) and the divergence theorem, from the principle of moment of momentum, at the last, we obtain

$$(11) \quad \sigma_{ij} - \sigma_{ji} + \varepsilon_{ijk} \rho (M_k + m_k) = 0.$$

Now, it remains to prescribe constitutive equations for $\mathbf{g}$, $\mathbf{m}$ and σ_{ij} . To this respect we can use the known technics.

Remarks. 1. The introduced entities, as internal body forces and internal couples, in order to describe the internal nonlocal body response to deformation, allow to drop out the so called localization residuals in the balance laws, used by some nonlocal theories.

2. If the internal body response is accompanied, in addition to our supposition, by internal surface couples, then Eqs. (11) become similar to the equations of a polar media. Indeed, let m_i and m_{ij} be the internal body couples and the couple stress tensor and let φ_i and $\chi_{ij} = \varphi_{i,j}$ be functions such that $m_i \varphi_i$ and $m_{ij} \chi_{ij}$ give the work of m_i respectively of m_{ij} per unit mass and unit time.

Then the indicated model will be a determinate one. When the internal body couples are absent we obtain the micropolar model [7].

3. The proposed model is convenient for heat conducting materials and for nonlocal magneto-thermoelasticity. In a next paper we will present some of these extensions.

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UN MEDIU CONTINUU NELOCAL CU TENSORUL TENSIUNE NESIMETRIC

(Rezumat)

Se consideră un mediu deformabil care răspunde la acțiunile exterioare prin forțe masice interioare nelocale și cupluri masice interioare nelocale. Se stabilesc ecuațiile de mișcare pentru un astfel de mediu.

THÉORÈMES VARIATIONNELS DANS LA THÉORIE DE LA VISCOÉLASTICITÉ LINÉAIRE ASSYMETRIQUE

PAR

D. MANOLE

0. Introduction. Dans cet article on présente deux théorèmes variationnels dans la théorie de la viscoélasticité linéaire assymétrique. Le Théorème 2.2. est l'analogue du théorème donné en [4] pour le cas de la viscoélasticité linéaire.

1. Équations de base. Soit un milieu viscoélastique qui occupe le domaine Ω de l'espace euclidien tridimensionnel, dont la fermeture est $\bar{\Omega}$ et la frontière (régulière) $\partial\Omega$. Les équations de base de la théorie de la viscoélasticité linéaire assymétrique sont :

1. Équations de mouvement

$$(1.1) \quad \begin{aligned} t_{ji,j} + F_i &= \rho u_i^{(2)}, \\ m_{ji,j} + e_{ijk} t_{jk} + M_i &= 0. \end{aligned} \quad (1.1)$$

2. Relations géométriques

$$(1.2) \quad \begin{aligned} \gamma_{ij} &= u_{j,i} - e_{kij} \omega_k, \\ \omega_{ij} &= \omega_{j,i}. \end{aligned} \quad (1.2)$$

3. Équations constitutives

$$(1.3) \quad \begin{aligned} t_{ij} &= t_{ij}^+ + [A_{ijkl} * \gamma_{kl} + B_{ijkl} * \omega_{kl}]^{(1)}, \\ m_{ij} &= m_{ij}^+ + [B_{klij} * \gamma_{kl} + C_{klij} * \omega_{kl}]^{(1)}. \end{aligned} \quad (1.3)$$

où

$$(1.4) \quad \begin{aligned} t_{ij}^+ &= \int_0^\infty A_{ijkl}^{(1)}(x, t+\tau) \gamma_{kl}^0(x, -\tau) d\tau + \int_0^\infty B_{ijkl}^{(1)}(x, t+\tau) \omega_{kl}^0(x, -\tau) d\tau, \\ m_{ij}^+ &= \int_0^\infty B_{klij}^{(1)}(x, t+\tau) \gamma_{kl}^0(x, -\tau) d\tau + \int_0^\infty C_{klij}^{(1)}(x, t+\tau) \omega_{kl}^0(x, -\tau) d\tau. \end{aligned} \quad (1.4)$$

4. Conditions sur la frontière

$$(1.5) \quad \begin{aligned} u_i &= \bar{u}_i \text{ sur } \partial\Omega_u \times [0, \infty), \quad \omega_i = \bar{\omega}_i \text{ sur } \partial\Omega_\omega \times [0, \infty), \\ t_i &\equiv t_{ji} n_j = \bar{t}_i \text{ sur } \partial\Omega_t \times [0, \infty), \quad m_i \equiv m_{ji} n_j = \bar{m}_i \text{ sur } \partial\Omega_m \times [0, \infty), \end{aligned}$$

$\bar{u}_i, \bar{\omega}_i, \bar{t}_i, \bar{m}_i$ étant des fonctions prescrites avec

$$\partial\Omega = \partial\Omega_u \cup \partial\Omega_t = \partial\Omega_\omega \cup \partial\Omega_m, \quad \partial\Omega_u \cap \partial\Omega_t = \partial\Omega_\omega \cap \partial\Omega_m = \emptyset$$

et les conditions initiales

$$(1.6) \quad \begin{aligned} u_i(x, 0) &= d_i(x), \quad x \in \bar{\Omega}, \\ \frac{\partial u_i}{\partial t}(x, 0) &= v_i(x), \quad x \in \bar{\Omega}; \end{aligned}$$

les notations sont celles de [1].

Les équations constitutives (1.4) peuvent être mises aussi sous la forme

$$(1.7) \quad \begin{aligned} t_{ij}(x, t) &= \dot{A}_{ijkl}(x) \gamma_{kl}(x, t) + \dot{B}_{ijkl}(x) \kappa_{kl}(x, t) + A_{ijkl}^{(1)} \star \gamma_{kl} + B_{ijkl}^{(1)} \star \kappa_{kl}, \\ m_{ij}(x, t) &= \dot{C}_{ijkl}(x) \gamma_{kl}(x, t) + \dot{D}_{ijkl}(x) \kappa_{kl}(x, t) + C_{ijkl}^{(1)} \star \gamma_{kl} + D_{ijkl}^{(1)} \star \kappa_{kl}. \end{aligned}$$

2. Théorèmes variationnels. Soit $s = \{u_i, \omega_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}\}$ un état admissible cinématique (voir [1]—[3]) et la fonctionnelle

$$(2.1) \quad \begin{aligned} \psi_t(x) &= \frac{1}{2} \int_{\Omega} l \star A_{ijkl} \star \gamma_{ij} \star \gamma_{kl} dx + \int_{\Omega} l \star B_{ijkl} \star \gamma_{ij} \star \kappa_{kl} dx + \\ &+ \frac{1}{2} \int_{\Omega} l \star C_{ijkl} \star \kappa_{ij} \star \kappa_{kl} dx - \int_{\Omega} g \star t_{ij} \gamma_{ij} dx - \int_{\Omega} g \star m_{ij}^+ \star \kappa_{ij} dx - \\ &- \int_{\Omega} g \star F_i \star u_i dx - \int_{\Omega} g \star M_i \star \omega_i dx - \int_{\partial\Omega_t} g \star \bar{t}_i \star u_i d\sigma - \int_{\partial\Omega_m} g \star \bar{m}_i \star \omega_i d\sigma, \end{aligned}$$

où

$$(2.2) \quad l(t) = 1, \quad g(t) = t, \quad t \in [0, \infty),$$

définie par la multitude K des états admissibles cinématiques.

Théorème 2.1 (Théorème du minimum de l'énergie potentielle au cas de l'équilibre viscoélastique). Si A_{ijkl}, C_{ijkl} sont symétriques par paires d'indices, s est une solution du problème mixte et

$$(2.3) \quad A_{ijkl} \star \gamma_{ij} \star \gamma_{kl} + 2B_{ijkl} \star \gamma_{ij} \star \kappa_{kl} + C_{ijkl} \star \kappa_{ij} \star \kappa_{kl} \geq 0$$

quel que soit γ_{ij} différent de zéro, alors

$$(2.4) \quad \psi_t(s) \leq \psi_t(s^*)$$

pour n'importe quelle $s^* \in K$.

Démonstration. Soit $s, s^* \in K$ et $s' = s^* - s$. Nous avons

$$(2.5) \quad \gamma'_{ij} = \gamma'_{ji} = e_{kij} \omega'_k, \quad \kappa'_{ij} = \omega'_{j,i},$$

$$(2.6) \quad \begin{aligned} t'_{ij} &= t_{ij}^* + [A_{ijkl} * \gamma'_{kl} + B_{ijkl} * x'_{kl}]^{(1)}, \\ m'_{ij} &= m_{ij}^* + [B_{kl ij} * \gamma'_{kl} + C_{ijkl} * x'_{kl}]^{(1)} \end{aligned}$$

et

$$(2.7) \quad u'_i = 0 \quad \text{sur} \quad \partial\Omega_u, \quad \omega'_i = 0 \quad \text{sur} \quad \partial\Omega_\omega.$$

Puisque

$$(2.8) \quad g * \frac{df}{dt} = l * f$$

si $f(0)=0$, la fonctionnelle $\psi_t(s^*)$, tenant compte de (1.3), on peut la mettre sous la forme

$$(2.9) \quad \begin{aligned} \psi_t(s^*) &= \psi_t(s) + \frac{1}{2} \int_{\Omega} l * A_{ijkl} * \gamma'_{ij} * \gamma'_{kl} dx + \int_{\Omega} l * B_{ijkl} * \gamma'_{ij} * x'_{kl} dx + \\ &+ \frac{1}{2} \int_{\Omega} l * C_{ijkl} * x'_{ij} * x'_{kl} dx + \int_{\Omega} g * t_{ij} * \gamma'_{ij} dx + \int_{\Omega} g * m_{ij} * x'_{ij} dx - \\ &- \int_{\Omega} g * F_i * u'_i dx - \int_{\Omega} g * M_i * \omega'_i dx - \int_{\partial\Omega_t} g * \bar{t}_i * u'_i d\sigma - \int_{\partial\Omega_m} g * \bar{m}_i * \omega'_i d\sigma. \end{aligned}$$

Remplaçant (2.5) en (2.9), appliquant le théorème de la divergente, tenant compte de (2.7) et groupant convenablement, on obtient

$$(2.10) \quad \begin{aligned} \psi_t(s^*) &= \psi_t(s) + \frac{1}{2} \int_{\Omega} l * A_{ijkl} * \gamma'_{ij} * \gamma'_{kl} dx + \int_{\Omega} l * B_{ijkl} * \gamma'_{ij} * x'_{kl} dx + \\ &+ \frac{1}{2} \int_{\Omega} l * C_{ijkl} * x'_{ij} * x'_{kl} dx - \int_{\Omega} g * (t_{ji,j} + F_i) * u'_i dx - \int_{\Omega} g * (m_{ji,j} + \\ &+ e_{ijk} t_{jk} + M_i) * \omega'_i dx + \int_{\partial\Omega_t} g * (t_{ji} n_j - \bar{t}_i) * u'_i d\sigma - \int_{\partial\Omega_m} g * (m_{ji} n_j - \bar{m}_i) * \omega'_i d\sigma. \end{aligned}$$

s étant solution du problème mixte, de (2.10) il résulte

$$(2.11) \quad \psi_t(s^*) - \psi_t(s) = \frac{1}{2} \int_{\Omega} l * (A_{ijkl} * \gamma'_{ij} * \gamma'_{kl} + 2B_{ijkl} * \gamma'_{ij} * x'_{kl} + C_{ijkl} * x'_{ij} * x'_{kl}) dx$$

et donc, sur la base de la condition (2.3), on peut écrire

$$(2.12) \quad \begin{aligned} \psi_t(s) &\leq \psi_t(s^*), \\ \psi_t(s) &= \psi_t(s^*) \Leftrightarrow \gamma'_{ij} = \gamma_{ij}^* - \gamma_{ij} = 0, \quad x'_{ij} = x_{ij}^* - x_{ij} = 0. \end{aligned}$$

Soit $s = \{u_i, \omega_i, \gamma_{ij}, x_{ij}, t_{ij}, m_{ij}\}$ un état admissible et la fonctionnelle

$$\begin{aligned}
 J_t(s) = & \frac{1}{2} [\rho u_m^{(1)}, u_m^{(1)}] + [t_{ji}, u_{ij} + e_{mji} \omega_m - \gamma_{ji}] - [F_m, u_m] + \\
 & + [m_{ji}, \omega_{ij} - x_{ji}] - [M_m, \omega_m] + \frac{1}{2} [A_{jikl}^{(1)} * \gamma_{kl} + A_{jikl}^0 \gamma_{kl}, \gamma_{ji}] + \\
 & + \frac{1}{2} [B_{jikl}^{(1)} * x_{kl} + \hat{B}_{jikl} x_{kl}, \gamma_{ji}] + \frac{1}{2} [B_{jikl}^{(1)} * \gamma_{kl} + \hat{B}_{jikl} \gamma_{kl}, x_{ji}] + \\
 & + \frac{1}{2} [C_{jikl}^{(1)} * x_{kl} + \hat{C}_{jikl} x_{kl}, x_{ji}] - [t_m, u_m - \bar{u}_m]_{\partial \Omega_u} - [m_m, \bar{\omega}_m - \\
 & - \omega_m]_{\partial \Omega_\omega} - [\bar{t}_m, u_m]_{\partial \Omega_t} - [\bar{m}_m, \omega_m]_{\partial \Omega_m} + [\rho(u_m^{(1)} - v_m), u_m]_0 - \\
 & - \frac{1}{2} [\rho(U_m - 2d_m), u_m^{(1)}]_0
 \end{aligned}
 \quad (2.13)$$

définie sur la multitude K des états admissibles, où

$$[f, h] = \int_{\Omega} \int_0^t f(x, \tau) h(x, t - \tau) d\tau dx, \quad (2.14)$$

$$[f, h]_0 = \int_{\Omega} \{f(x, \tau) h(x, t - \tau)\} |_{\tau=0} dx, \quad (2.15)$$

$$[f, h]_{\partial \Omega} = \int_{\partial \Omega} \int_0^t f(x, \tau) h(x, t - \tau) d\tau d\sigma. \quad (2.16)$$

Théorème 2.2. Soit K la multitude des états admissibles et J une fonctionnelle définie sur K par (2.13) pour n'importe quelle $s = \{u_j, \omega_i, \gamma_{ij}, x_{ij}, t_{ij}, m_{ij}\} \in K$. Alors

$$\delta J_t(s) = 0, \quad (2.17)$$

à condition que et seulement à condition que s soit une solution du problème mixte.

Démonstration. Soit $s = \{u_i, \omega_i, \gamma_{ij}, x_{ij}, t_{ij}, m_{ij}\}$ et $s' = \{u'_i, \omega'_i, \gamma'_{ij}, x'_{ij}, t'_{ij}, m'_{ij}\}$ deux états admissibles. Évidemment $s + \lambda s' \in K$ pour n'importe quel scalaire λ . Si on forme l'expression

$$\delta_s J_t(s) = \frac{d}{d\lambda} J_t(s + \lambda s')|_{\lambda=0}$$

on déduit que

$$\begin{aligned}
 \delta_s J_t(s) = & [\rho u'_m, u'_m] + [t'_{ji}, u'_{ij} + e_{mji} \omega'_m - \gamma'_{ji}] + [u'_{ij} + e_{mji} \omega_m - \gamma_{ji}, t'_{ji}] + \\
 & + [m'_{ji}, \omega'_{ij} - x'_{ji}] + [\omega'_{ij} - x_{ji}, m'_{ji}] - [F_m, u'_m] - [M_m, \omega'_m] + \\
 & + [A_{jikl}^{(1)} * \gamma_{kl} + B_{jikl}^{(1)} * x_{kl} + \hat{A}_{jikl} \gamma_{kl} + \hat{B}_{jikl} x_{kl}, \gamma'_{ji}] +
 \end{aligned}$$

$$\begin{aligned}
 (2.18) \quad & + [B_{klij}^{(1)} \times \gamma_{kl} + C_{jikl}^{(1)} \times z_{kl} + \dot{B}_{klij} \gamma_{kl} + \dot{C}_{jikl} z_{kl}, z'_{ji}] - \\
 & - [u_m - \bar{u}_m, t'_m]_{\partial\Omega_u} - [\bar{t}_m, u'_m]_{\partial\Omega_t} - [\bar{\omega}_m - \omega_m, m'_m]_{\partial\Omega_\omega} - \\
 & - [\bar{m}_m, \omega'_m]_{\partial\Omega_m} + [\rho(u_m^{(1)} - v_m), u_m]_0 + [\rho u_m^{(1)}, u_m]_0 - \\
 & - \frac{1}{2} [\rho(u_m - 2d_m), u_m^{(1)}]_0 - \frac{1}{2} [\rho u'_m, u_m^{(1)}]_0.
 \end{aligned}$$

Intégrant par parties et tenant compte du théorème de la divergente, on obtient

$$\begin{aligned}
 (2.19) \quad & \delta_s J(s) = [u_{ij} + e_{mji} \omega_m - \gamma_{ji}, t'_{ji}] + [\omega_{i,j} - z_{ji}, m'_{ji}] - \\
 & - [t_{ji,j} + F_i - \rho u_i^{(2)}, u'_i] - [m_{ji,j} + e_{ijm} t_{jm} + M_i, \omega'_i] + \\
 & + [A_{jikl}^{(1)} \times \gamma_{kl} + B_{jikl}^{(1)} \times z_{kl} + \dot{A}_{jikl} \gamma_{kl} + \dot{B}_{jikl} z_{kl} - t_{ji}, \gamma'_{ji}] + \\
 & + [B_{klij}^{(1)} \times \gamma_{kl} + C_{jikl}^{(1)} \times z_{kl} + \dot{B}_{klij} \gamma_{kl} + \dot{C}_{jikl} z_{kl} - m_{ji}, z'_{ji}] - \\
 & - [u_m - \bar{u}_m, t'_m]_{\partial\Omega_u} + [\bar{t}_m - t_m, u'_m]_{\partial\Omega_t} - [\omega_m - \omega'_m, m'_m]_{\partial\Omega_\omega} + \\
 & + [\bar{m}_m - m_m, \omega'_m]_{\partial\Omega_m} + [\rho(u_m^{(1)} - v_m), u'_m]_0 - [\rho(u_m - d_m), u_m^{(1)}]_0.
 \end{aligned}$$

Si s est une solution du problème mixte, alors de (2.19) il résulte que

$$(2.20) \quad \delta_s J_t(s) = 0$$

pour n'importe quelle $s' \in K$ et donc (2.17) a lieu ($\delta_s J_t(s)$ étant linéaire en s').

Pour démontrer la suffisance on suppose qu'a lieu la relation (2.17) qui implique (2.20). Il est évident que $\Lambda = \{0, 0, 0, 0, 0, 0\}$ et donc on peut choisir $\lambda' = \{u'_i, 0, 0, 0, 0, 0\}$ de sorte que u'_i avec ses dérivées spatiales s'annule sur $\partial\Omega_u \times [0, \infty)$ et sur Ω pour $t=0$. Donc (2.19) et (2.20) implique

$$(2.21) \quad \int_{\Omega} \{t_{ji,j} + F_i - \rho u_i^{(2)}\} \times u'_i dx = 0, \quad t \geq 0$$

et, se basant sur le lemme fondamental du calcul variationnel [5], il résulte

$$(2.22) \quad t_{ji,j} + F_i - \rho u_i^{(2)} = 0.$$

Considérant le même choix pour s' mais imposant que u soit zéro sur $\partial\Omega_t \times [0, \infty)$, alors (2.19), (2.20), (2.22) et le lemme fondamental impliquent

$$(2.23) \quad m_{ji,j} + e_{ijm} t_{jm} + M_i = 0.$$

Choisissant $\Lambda' = \{0, \omega'_i, 0, 0, 0, 0\}$ et imposant les mêmes conditions que pour u'_i , on obtient (1.2).

Continuant ainsi le raisonnement il résulte la démonstration du théorème.

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TEOREME VARIAȚIONALE ÎN TEORIA VISCOELASTICITĂȚII LINIARE ASIMETRICE

(Rezumat)

Se prezintă două teoreme variaționale în teoria viscoelasticității liniare asimetrice; teorema 2.2 este analoga teoremei dată în [4] în cazul viscoelasticității liniare.

ON THE UNIQUENESS OF SOME FLOWS IN THE LINEAR THEORY OF HEAT CONDUCTING, INCOMPRESSIBLE SIMPLE FLUIDS

BY

C. FETECĂU

0. In [1] it was proved the uniqueness of some linear and bilinear flows of incompressible fluids of the integral type. It is the aim of this paper to extend those results to more general fluids.

The constitutive equations of a linear, heat conducting, incompressible simple fluid with memory, as they were deduced in [2], are

$$\begin{aligned} \mathbf{T} &= \mathbf{S} + p\mathbf{1} = -2\rho \int_0^\infty \zeta'(s) \mathbf{G}(s) ds, \quad \mathbf{G} = \mathbf{F}_t'^T \mathbf{F}_t' - \mathbf{1}, \\ (1) \quad \mathbf{q} &= -k(0)\mathbf{g} - \int_0^\infty k'(s) \mathbf{g}^t(s) ds, \quad \eta = -\alpha(0), \quad \varepsilon = \psi - \theta\eta, \\ \psi &= \psi_0 + \alpha(0)\theta + \int_0^\infty \alpha'(s) \theta^t(s) ds + \int_0^\infty \zeta'(s) \text{tr} \mathbf{G}(s) ds, \end{aligned}$$

where $\mathbf{T}$, $\mathbf{S}$, $\mathbf{q}$, η , ψ and ε are, respectively, the extra stress tensor, the stress tensor, the heat flux, the entropy, the free energy and the internal energy at time t ; $\mathbf{F}_t'$, θ^t and $\mathbf{g}^t$ are the relative deformation gradient, the temperature and the temperature gradient histories; ζ , k and α are scalar material functions; ρ is the density, p is an arbitrary constant and ψ_0 is also a constant.

1. The types of flows to be firstly considered, have the velocity field $\dot{\mathbf{x}}(\mathbf{x}, t)$ in a Cartesian co-ordinate system x, y, z

$$(2) \quad \dot{x} = 0, \quad \dot{y} = u(x, t), \quad \dot{z} = v(x, t).$$

Assuming θ to be a function of x and t only, we can easily prove that the motion and energy balance equations reduce to (for (3)₁₋₂ see [1] where the body force $\mathbf{b}$ is conservative)

$$\begin{aligned}
 & 2\rho \int_0^\infty \zeta'(s) \int_{t-s}^t \partial_x^2 u(x, \sigma) d\sigma ds + a(t) = \rho \partial_t u(x, t), \\
 & 2\rho \int_0^\infty \zeta'(s) \int_{t-s}^t \partial_x^2 v(x, \sigma) d\sigma ds + b(t) = \rho \partial_t v(x, t), \\
 (3) \quad & \int_0^\infty k'(s) \partial_x^2 \theta(x, t-s) ds + k(0) \partial_x^2 \theta(x, t) - \rho \int_0^\infty \alpha'(s) \partial_t \theta(x, t-s) ds - \\
 & - \rho \int_0^\infty \zeta'(s) \partial_t \operatorname{tr} \mathbf{G}(s) ds + S_{12} \partial_x u + S_{13} \partial_x v = 0.
 \end{aligned}$$

In the following we shall prove that if these equations with boundary and initial conditions have a solution, then it is unique.

For this, let us imagine a bilinear flow between two infinite parallel plates *I* and *II* which till $t=0$ they had been in rest and after $t=0$ they had moved with the speeds $U(t)$ and $V(t)$ in the directions y and z , respectively. Assuming that the fluid adheres to plates, we can write the next conditions for u, v and θ

$$\begin{aligned}
 & u(x, t) = v(x, t) = \theta(x, t) \text{ on } [0, d] \times (-\infty, 0], \\
 (4) \quad & \left. \begin{aligned} u(0, t) = 0, \quad u(d, t) = U(t) \\ v(0, t) = V(t), \quad v(d, t) = 0 \\ \theta(0, t) = \theta(d, t) = 0, \end{aligned} \right\} \text{ for } t \in (0, \infty),
 \end{aligned}$$

where d is the distance between the plates.

Looking at the relations (3), (4) we observe that our problem reduces to a system of two differentiable and functional equations in u, v and to a coupled equation in u, v and θ . As, the system in u and v is similar to that who was studied in [1] we shall stopped here only to the equation in θ , assuming that u and v are unique determined.

Theorem 1. *If the equation (3), with initial and boundary conditions from (4), has a continuous on $[0, d] \times (-\infty, \infty)$ then it is unique.*

In order to prove this, we shall use again the well known theorem of Steklov [3], Ch. 4, § 1 (see also [1]). Let $\theta(x, t)$ be a continuous solution of our problem. We can write it as a Fourier serie absolutely and uniformly convergent in terms of the eigenfunctions of the problem

$$(5) \quad \partial_x^2 \theta(x, t) + \lambda \theta(x, t) = 0, \quad \theta(0, t) = \theta(d, t) = 0,$$

i.e.,

$$(6) \quad \theta(x, t) = \sum_1^\infty \theta_n(t) \sin \frac{n\pi}{d} x, \quad \theta_n(t) \equiv 0 \text{ on } (-\infty, 0].$$

Now, introducing (6) in (3), then multiplying by $\sin \frac{n\pi}{d} x$ and integrating from 0 to d we get

$$(7) \quad \frac{\lambda_n d}{2} \int_0^\infty k'(s) \theta_n(t-s) ds - \frac{d \lambda_n k(0)}{2} \theta_n(t) - \rho \frac{d}{2} \int_0^\infty \alpha'(s) \partial_t \theta_n(t-s) ds + f_n(t) = 0,$$

$$f_n(t) = \int_0^d r(x, t) \sin \frac{n\pi}{d} x dx, \quad \lambda_n = \frac{n^2 \pi^2}{d^2}.$$

Instituting a change of variables ($\tau = t - s$), using (6)₂ and then integrating by parts the first integral from (7) we attain

$$(8) \quad \theta_n(t) = g_n(t) + \int_0^t \gamma_n(t - \tau) \theta_n(\tau) d\tau, \quad g_n(0) = 0,$$

where

$$g_n(t) = \frac{2}{d} \frac{f_n(t)}{\rho \alpha'(0) + \lambda_n k(0)} \quad \text{and} \quad \gamma_n(t) = \frac{\lambda_n k'(t) - \rho \alpha''(t)}{\rho \alpha'(0) + \lambda_n k(0)}.$$

It is clear now, that the problem of uniqueness of the solution of (3)₃ reduces to that of the equation (8), or this equation has a unique solution on $[0, \infty)$ (see [4], p. 75, problem 4) if $g_n(t)$ is continuous on $[0, \infty)$ and $k_n(t)$ is measurable on each interval $[0, T]$.

2. Let us consider now a lineal flow whose velocity field is

$$(9) \quad \dot{x} = 0, \quad \dot{y} = v(x, z, t), \quad \dot{z} = 0.$$

A such flow can be obtained between four walls which are at rest. For its determination we must solve

$$(10) \quad \begin{aligned} & 2\rho \int_0^t \zeta'(s) \int_{t-s}^t \nabla v(x, z, \sigma) d\sigma ds + a(t) = \rho \partial_t v(x, z, t), \\ & \int_0^\infty k'(s) \nabla \theta(x, z, t-s) ds + k(0) \nabla \theta(x, z, t) - \rho \int_0^\infty \alpha'(s) \partial_t \theta(t-s) ds - \\ & - \rho \int_0^\infty \zeta'(s) \partial_t \operatorname{tr} \mathbf{G}(s) ds + S_{12} \partial_x v + S_{23} \partial_z v = 0, \end{aligned}$$

where

$$(11) \quad \left. \begin{aligned} v(x, z, t) &= \theta(x, z, t) = 0 \quad \text{on} \quad [0, d] \times [0, h] \times (-\infty, 0], \\ v(0, z, t) &= v(d, z, t) = v(x, 0, t) = v(x, h, t) = 0 \\ \theta(0, z, t) &= \theta(d, z, t) = \theta(x, 0, t) = \theta(x, h, t) = 0 \end{aligned} \right\} t \in (0, \infty).$$

As a similar equation to $(10)_1$, was studied in [1] we shall again keep our attention only to $(10)_2$.

Theorem 2. *If the equation $(10)_2$ with the corresponding initial and boundary conditions has a continuous solution $\theta(x, z, t)$ on $[0, d] \times [0, h] \times (-\infty, \infty)$ then it is unique.*

The Steklov's theorem enable again us to write $\theta(x, z, t)$ for each $t > 0$ as a Fourier series absolutely and uniformly convergent in terms of the eigenfunctions of the problem

$$(12) \quad \nabla \theta + \lambda \theta = 0, \quad \theta|_{x=0} = \theta|_{x=d} = \theta|_{z=0} = \theta|_{z=h} = 0,$$

i.e.,

$$(13) \quad \theta(x, z, t) = \sum_{m,n=1}^{\infty} \theta_{mn}(t) \sin \frac{n\pi}{d} x \sin \frac{m\pi}{h} z, \quad \theta_{mn}(0) = 0.$$

Similar accounts as in the first theorem lead us to the next equation for $\theta_{mn}(t)$,

$$(14) \quad \theta_{mn}(t) = g_{mn}(t) + \int_0^t \gamma_{mn}(t - \tau) \theta_{mn}(\tau) d\tau, \quad \theta_{mn}(0) = 0$$

where $g_{mn}(t)$ and $\gamma_{mn}(t)$ are known functions. As this equation is of the same form as (8) we can consider our theorem as proved.

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ASUPRA UNICITĂȚII UNOR CURGERI ÎN TEORIA LINIARĂ A FLUIDELOR SIMPLE INCOMPRESIBILE, CONDUCĂTOARE DE CĂLDURĂ

(Rezumat)

Se demonstrează unicitatea unor curgeri liniare și biliniare ale fluidelor simple incompresibile, conducătoare de căldură, când ecuațiile constitutive considerate sînt liniarizate. Se folosește în acest scop dezvoltarea în serie Fourier după valorile proprii ale unei probleme la limită.

TENSOR OPERATORS AND NONEUCLIDEAN SPACES

BY

I. GOTTLIEB and GH. ZET

1. Introduction. The idea of definition of operators with tensorial character has appeared early in quantum mechanics. It is known that, in quantum mechanics, to each physical observable corresponds a linear Hermitian operator [1]. If the physical observable is a vector, then the corresponding operator has, evidently, a vectorial character; therefore, in this case, we have a vectorial operator. Generally, all observable physical quantities correspond to tensor operators.

At first sight, it can appear that the definition and the study of the tensor operators will not be giving new results, unknown in quantum mechanics or, in general, in the operators theory. However, there are many interesting properties of the tensor operators that are not known yet. We can illustrate this fact, considering here the well known example of the momentum operator

$$(1) \quad \hat{p} = \frac{\hbar}{i} \nabla,$$

where the operator ∇ has the components $\partial/\partial x^i$ with respect to the rectangular cartesian coordinate system x^i ($i=1, 2, 3$). If we pass to an arbitrary curvilinear coordinate system, then the momentum components transform like those of a vector. Therefore, we obtain the components co- and contravariant of the momentum, $\hat{p}_i$ and respectively $\hat{p}^i$. These components are connected by the relation

$$(2) \quad \hat{p}_i = g_{ik} \hat{p}^k,$$

where g_{ik} is the fundamental metric tensor of three-dimensional euclidean space with the curvilinear coordinate system x^i . The operator ∇ becomes the operator of covariant derivative ∇_k and we have

$$(3) \quad \hat{p}_j = \frac{\hbar}{i} \nabla_j, \quad \hat{p}^j = \frac{\hbar}{i} g^{jk} \nabla_k \quad (i, k=1, 2, 3).$$

Therefore, we are confronted with problem of generalization of this result, about rising of indices, to the arbitrary tensor operators. This generalization imply, of course, the use of the differential geometry formalism.

Furthermore, the extension of the axioms of linear operators to the tensor operator case is not trivial. As an example, let us suppose that $\hat{V}_i$ is a vectorial operator. Let us define the linearity of this operator by the ordinary conditions:

$$(4) \quad \hat{V}_i(f+g) = \hat{V}_i f + \hat{V}_i g,$$

$$(5) \quad \hat{V}_i(cf) = c(\hat{V}_i f),$$

where f and g are the elements of the space over that the operator $\hat{V}_i$ is defined and c is a constant. Let us suppose now that the vector A_i has the property that $A_i = \text{const.}$ with respect to some curvilinear coordinate system. We can not pick A_1 as c in (5) because

$$(6) \quad \nabla_i A_1 = -\Gamma_{ii}^1 A_1 \neq 0.$$

Therefore, the constant c from Eq. (5) is a scalar constant.

The condition (5) imply, in the same time, that contravariant components of $\hat{V}$ are linear too. It is necessary, therefore, to generalise conveniently this condition.

In this work we give some general properties of the tensor operators. For generality, we will consider the noneuclidean spaces case.

2. Definition of Tensor Operators. Let V_n be a n -dimensional manifold of class C^k and let us note by x^i ($i=1,2,\dots,n$) a local coordinate system on this manifold. The tangent space to V_n in the point x will be noted by T_x and its dual by T_x^* . The set of all tensors on V_n forms an affine tensorial algebra, $\mathcal{T}$, over the space x [2]. This algebra has, among others, two subalgebras: the contravariant tensorial algebra $T(T_x)$ and the covariant tensorial algebra $T(T_x^*)$.

If $\{e_i\}$ is a basis in T_x and $\{\theta^i\}$ a basis in T_x^* , then the basis of the affine tensorial algebra $\mathcal{T}$ is given by tensorial products of e_i and θ^i . For example, if T is a tensor of $(1,2)$ type, i.e. from the space $T_x \otimes T_x^* \otimes T_x^*$, then we have the decomposition

$$(7) \quad T = T_{ijk}^i e_i \otimes \theta^j \otimes \theta^k.$$

Let us note by $\mathcal{T}_q^p$ the set of all tensors of (p,q) type; this set forms a vectorial space over the real number set $\mathbb{R}$. (For applications in physics it is necessary to generalise these results to the case when the tensorial components are complex functions on real variables).

Definition 1. A tensor operator $\hat{O}$ is an application of $\mathcal{T}$ algebra onto itself

$$(8) \quad \hat{O} : \mathcal{T} \rightarrow \mathcal{T}.$$

Definition 2. The tensor operator $\hat{O}$ is named of (p,q) type, if

$$(9) \quad \hat{O} : \mathcal{T}_i^r \rightarrow \mathcal{T}_{i+q}^{r+p}.$$

If $\hat{O}$ is a tensor operator of (p, q) type, then we will note its components by

$$(10) \quad \hat{O}_{j_1 j_2 \dots j_p}^{i_1 i_2 \dots i_p} = \hat{O}_{(j)}^{(i)}.$$

Then, from the definition 2, it results that

$$(11) \quad \hat{O}_{(j)}^{(i)}[T_{(l)}^{(k)}] = T_{(l)(j)}^{(k)(i)}.$$

3. The Transformation Law of Tensor Operators. Making use of the relation (11), we may obtain the transformation law of a tensor operator with respect to the coordinates change:

$$(12) \quad \bar{x}^i = \bar{x}^i(x^k).$$

Let us introduce, first, the notations

$$(13) \quad \bar{x}_i^r = \frac{\partial \bar{x}^r}{\partial x^i}; \quad \underline{x}_p^k = \frac{\partial x^k}{\partial \bar{x}^p}; \quad \bar{x}_{(i)}^{(r)} = \bar{x}_{i_1}^{r_1} \bar{x}_{i_2}^{r_2} \dots \bar{x}_{i_n}^{r_n}, \quad \text{etc.}$$

Then we have

$$(14) \quad T_{(l)}^{(k)} = \underline{x}_{(p)}^{(k)} \bar{x}_{(l)}^{(q)} \bar{T}_{(q)}^{(p)}$$

and

$$(15) \quad T_{p(l)}^{(k)(i)} = \underline{x}_{(p)}^{(k)} \bar{x}_{(l)}^{(q)} \underline{x}_{(r)}^{(i)} \bar{x}_{(j)}^{(s)} \bar{T}_{(q)(s)}^{(p)(r)}.$$

Replacing Eqs. (14) and (15) in Eq. (11) we obtain

$$(16) \quad \hat{O}_{(j)}^{(i)} \underline{x}_{(p)}^{(k)} \bar{x}_{(l)}^{(q)} \bar{T}_{(q)}^{(p)} = \underline{x}_{(r)}^{(i)} \bar{x}_{(j)}^{(s)} \underline{x}_{(p)}^{(k)} \bar{x}_{(l)}^{(q)} \bar{T}_{(q)(s)}^{(p)(r)}.$$

Finally, multiplying this equality with

$$\bar{x}_{(i)}^{(m)} \underline{x}_{(n)}^{(j)} \bar{x}_{(k)}^{(u)} \underline{x}_{(v)}^{(l)}$$

on the left side, and summing over $(i), (j), (k)$ and (l) indices, we obtain

$$(17) \quad \underline{x}_{(i)}^{(m)} \bar{x}_{(n)}^{(j)} \bar{x}_{(k)}^{(u)} \underline{x}_{(v)}^{(l)} \hat{O}_{(j)}^{(i)} \underline{x}_{(p)}^{(k)} \bar{x}_{(l)}^{(q)} \bar{T}_{(q)}^{(p)} = \bar{T}_{(v)}^{(u)} \underline{x}_{(n)}^{(m)}.$$

From this equation it results easy the transformation law of the components of tensor operators $\hat{O}$:

$$(18) \quad \hat{\bar{O}}_{(s)}^{(r)} = \bar{x}_{(i)}^{(r)} \bar{x}_{(k)}^{(p)} \underline{x}_{(s)}^{(j)} \underline{x}_{(q)}^{(l)} \hat{O}_{(j)}^{(i)} \underline{x}_{(p)}^{(k)} \bar{x}_{(l)}^{(q)}.$$

We can see that the transformation law of tensor operators depends on the tensorial character of the quantities over that they are mapped.

4. Linear Tensor Operators. An important class of tensor operators, from both mathematical and physical viewpoint, is that of linear tensor operators. In order to define the linear tensor operators, we suppose now that V_n is a space with affine connexion Γ . In all that follows we will note by $\mathbb{O}$ the set of the tensors over V_n having the property

$$(19) \quad \nabla_k L_{(j)}^{(i)} = 0, \quad L_{(k)}^{(i)} \in \mathbb{O},$$

where ∇_k denotes the covariant derivative with respect to the connexion Γ .

If $T^{(r)}_{(s)}$ is an arbitrary tensor on V_n , then we have

$$(20) \quad \nabla_k [L^{(i)}_{(j)} T^{(r)}_{(s)}] = L^{(i)}_{(j)} \nabla_k T^{(r)}_{(s)}.$$

Based on this property, we will define the linear tensor operators as follows.

Definition 3. A tensor operator $\hat{O}$ is named linear if the following two axioms are satisfied:

$$(21) \quad \hat{O}(T+U) = \hat{O}T + \hat{O}U, \quad T, U \in \mathcal{T},$$

$$(22) \quad \hat{O}(LT) = L\hat{O}T, \quad L \in \mathbb{O}.$$

In these relations we omitted the indices, but their introduction is evidently.

In particular, if V_n is a metric manifold, g_{ik} being its metric, we have

$$(23) \quad \nabla_s g_{ik} = 0,$$

and therefore $g \in \mathbb{O}$. As examples of such connexions we can consider the Christoffel connexion, the parallelizable connexion [3] etc. We observe, therefore, that we can rise the tensorial indices without modifying the linearity.

The axiom (22) for the constants of $\mathcal{T}$ algebra includes the axiom (5) for the scalar constants. Indeed, if c is a constant scalar, then $\nabla_k c = \partial c / \partial x^k = 0$ and therefore $c \in \mathbb{O}$.

If we try to obtain some others important properties of the tensor operators, it is necessary to introduce an inner product on $\mathcal{T}$ algebra. This will be done in a next paper.

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OPERATORI TENSORIALI ȘI SPAȚII NEEUCLIDIENE

(Rezumat)

Se definesc operatorii tensoriali în general, precum și operatorii tensoriali liniari ca un caz important atît din punct de vedere matematic cit și fizic. Se pune în evidență faptul că studiul operatorilor tensoriali poate conduce la proprietăți noi față de rezultatele cunoscute în prezent în mecanica cuantică sau teoria operatorilor.

THE METRIC OF A ROTATING SYSTEM OF REFERENCE OBTAINED WITHOUT THE FIELD EQUATIONS AND A PHYSICAL SIGNIFICANCE OF THE PROPER ACCELERATION IN GENERAL RELATIVITY

BY

CORNELIU D. CIUBOTARIU

In some recent papers [1], [2] we have dealt with the problem of deriving general relativistic line-elements without the field equation. Our method differs from that of the other authors (e.g. [3]—[5]) by the use of a tetrad technique and a Newtonian interpretation of the acceleration field. We have obtained by this way the Schwarzschild solution and the metric of a uniform field, both metrics corresponding to an exterior region, outside the matter.

In this Note, we shall obtain the K u r s u n o g l u's metric [6] using the same method but, now, it refers to an interior region, inside a rotating fluid.

We consider a rotating fluid with cylindrical symmetry about the axis of rotation and with a constant angular velocity ω . The most general form of the corresponding metric, by suitable transformations of coordinates (without losing of generality), in cylindrical coordinates (ct, r, θ, z) can be written as

$$(1) \quad ds^2 = F(r)c^2 dt^2 - dr^2 - G(r)d\theta^2 - dz^2,$$

where F and G are functions of r alone. A particle with fixed spatial coordinates in this metric has the 4-velocity

$$(2) \quad u = (F^{-\frac{1}{2}}, 0, 0, 0),$$

and the frame vectors along the coordinates axes are

$$(3) \quad \lambda_{(\alpha)}^\mu = \delta_\alpha^0 \delta_0^\mu F^{-\frac{1}{2}} + \delta_\alpha^1 \delta_1^\mu + \delta_\alpha^2 \delta_2^\mu G^{-\frac{1}{2}} + \delta_\alpha^3 \delta_3^\mu,$$

with

$$\lambda_{(0)}^\mu = u^\mu.$$

Using the metric (1), the only non-vanishing Christoffel symbols are

$$(4) \quad \Gamma_{22}^1 = -\frac{1}{2} G', \quad \Gamma_{00}^1 = \frac{1}{2} F', \quad \Gamma_{12}^1 = \frac{1}{2} (\ln G)', \quad \Gamma_{10}^0 = \frac{1}{2} (\ln F)',$$

where the primes denote differentiation with respect to τ . From the Frenet-Serret formulas [7] we obtain

$$(5) \quad a^\nu = \frac{\delta \lambda^\nu(0)}{\delta_s} = b \lambda_{(1)}^\nu,$$

where b is the first curvature of the world line of the particle. By the use of the equivalence principle we may identify a^ν with the 4-acceleration of the particle and the magnitude of a^ν with the proper acceleration α/c^2 (see [8]).

In the following we consider the case of small velocities ($\omega^2 r^2/c^2 \ll 1$). A fixed particle of the rotating system (3) will experience the proper acceleration $\alpha_i = \omega^2 r$, the indice i referring to the word „inertial“. The gravitational equivalent acceleration which compensates the inertial field α_i and transforms the fixed particle in a „freely falling particle“ is

$$(6) \quad \alpha = -\omega^2 r.$$

Now we can calculate the components of the metric tensor. From (4) and (5) we have

$$(7) \quad a^\nu = (0, \Gamma_{00}^1 u^0, u^0, 0, 0) = \left(0, \frac{1}{2} \frac{F'}{F} 0, 0\right)_s,$$

and from (6), we get

$$(8) \quad \frac{1}{2} \frac{F'}{F} = -\frac{\omega^2}{c^2}.$$

If we suppose for $\omega=0$, the relation (1) represents the Lorentz metric, from (8) we find

$$(9) \quad g_{00} \equiv F = 1 - \frac{\omega^2 r^2}{c^2}.$$

Now it is easily to obtain g_{11} from the Tangherlini's normalization condition [5],

$$(10) \quad g_{00} g_{22} = -r^2,$$

or from the kinematic relations of the Lorentz transformations [3]. Finally, we have the Kursunoglu's metric

$$(11) \quad ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2 dt^2 - dr^2 - \frac{r^2}{1 - \frac{\omega^2 r^2}{c^2}} d\theta^2 - dz^2.$$

For the sake of comparison we give brief mention to the Irvine's metric [9] of a rotating system of reference, with the z axis as the axis of rotation

$$(12) \quad ds^2 = dr^2 + r^2 d\theta^2 + dz^2 + \frac{2\omega r^2}{c} d\theta c dt - \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2 dt^2.$$

This metric has been obtained by a coordinate transformation and therefore it describes a flat space-time. The metric (11) is associated to a curved space-time, the Riemann-Christoffel tensor $R^{\alpha}_{\beta\gamma\delta}$ being not zero.

A question arises: what is the metric which is correctly to be used for the study of the various distributions of matter and fields? We note only that, for the same problem, I. Walk and R.C.T. da Costa [10] make use of the line element (11), whereas Irvine [9] applies the metric (12) to discuss the electrodynamics in a rotating system of reference.

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METRICA UNUI SISTEM DE REFERINȚĂ ÎN ROTAȚIE OBTINUTĂ FĂRĂ ECUAȚIILE DE CÎMP ȘI O SEMNIFICAȚIE FIZICĂ A ACCELERAȚIEI PROPRII ÎN RELATIVITATEA GENERALĂ

(Rezumat)

Metoda teoretică în cazul unui fluid cu simetrie cilindrică este aplicată pentru a obține metrica unui sistem de referință în rotație fără a folosi ecuațiile lui Einstein. Se arată că principiul echivalenței dă o semnificație fizică a accelerației proprii în relativitatea generală.

THE EFFECTS OF MECHANICAL STRESS ON LIQUID CRYSTAL DEVICES

BY

H. N. TEODORESCU E. SOFRON, I. BACIU, L. ȘCHIOPU and E. LUCA

1. Introduction. The great sensitivity of the nematic and cholesteric mesophases with respect to various physical parameters is well known and has important implications in their use.

The influence of the mechanical pressure and deformation of the nematic liquid crystal devices (LCD) are less studied in the literature, even if possible applications of the induced phenomena are predicted and experimented [1]—[4].

The aim of this paper is to contribute to the detailed and systematic investigation of these effects in order to understand the physical processes developing in the liquid crystal layer and to create the experimental basis which is necessary in developing applications.

2. Experimental Investigation of the Induced Phenomena. 2.1. *Experimental Method.* Two types of LCD were investigated: working by field effect and by dynamic scattering effect. The investigation was performed: i) microscopically, in polarized and unpolarized light; ii) macroscopically, optical, with the device described below and iii) electrically.

In order to perform the macroscopic, punctual investigation, an experimental device based on optical fibers was realised. The device has three parts. The mechanics is based on a micrometric table which allows the displacement of the probe (LCD) on two orthogonal directions with a precision of 0,1 mm. The optics contains a light source and a testing head realised with a bundle of multimode optical fibers having low numerical aperture. The electronics contains a photosensor excited by the optical fibers of the testing head, a resistive bridge supplied by a signal generator (10 kHz), a measuring amplifier and an electronic voltmeter or a X—Y recorder (see Fig. 1a and 1b).

The electrical investigation was performed in an experimental arrangement including a signal generator supplying variable amplitude signals (0...25 V), a resistance bridge and a X—Y recorder. The displacements of the glass walls of the LCD was measured with a microcomparator with sensibility greater than 5 μm .

2.2. *Optical Investigation.* The microscopical investigation of the LCD was performed both in polarized and unpolarized light. Two types of changes of the microstructure are noticed in unpolarized light and they are due to

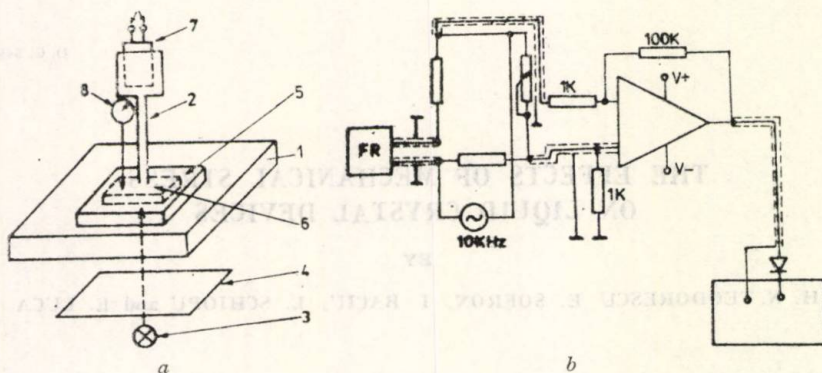


Fig. 1. — a) The mechanics and the optics of the testing device. 1—micrometric table ; 2—testing head with optical fibers ; 3—light source ; 4—polarizer ; 5—LCD ; 6—analizer ; 7—photoresistance ; 8—microcomparator ; F—exercited force. b) The electronics of the testing device.

the same effect : the flow in the liquid crystal layer. In the absence of the electric field, the mobility and the dimensions of the domains created by the defects of the texture are correlated to the direction and the velocity of the flow induced by the pressure applied to the LCD. It was also remarked the tendency of generating new texture defects which are travelling in the crystal. The photo reproduced in Fig. 2 ($\times 200$) shows these defects. The phenomena were observed both for LCD based on dynamic scattering of the light and on field effect. The OFF time associated to these effects is great (seconds).

In polarized light, in the absence of the electric field or for voltages under the threshold, the phenomena occurring as result of the mechanical stress are more evident. The initial texture of the investigated dynamic scattering mode LCD changes fasty, the time constants of these effect being under 1 second. The changes are much more significant and the effect more sensitive than under unpolarized light.

Macroscopically, in polarized light, the changes of the optical transmittence of the field effect LCD are also visible. Using field effect LCDs and the experimental arrangement indicated in Fig. 1 it was found that the sensitivity to the displacements of the wall of the LCD is better than $5 \mu\text{m}$. The recordings indicate that there is no proportionality between the optical effect and the amplitude of the displacement. The change of the transmittance in polarized light is dependent rather on the velocity of the displacement than on the amplitude. This may be explained if the texture changes are determined in the first place by the flow velocity, not by the flow amplitude. Using in the arrangement indicated in Fig. 1 a X—Y recorder, curves as those plotted in Fig. 3 are obtained.



Fig. 2. — Microphotograph ($\times 200$) of the pressure induced defects in LCD.

2.3. The Electric Investigation of the LCD. In the experiments described below there are observed the changes of the impedance of the devices under the effect of mechanical stresses. The experimental arrangement contains a constant voltage generator (50 Hz), a resistive bridge and a X—Y recorder. For both types of LCD, changes of the impedance were obtained, most important for the dynamic scattering effect LCDs. The electrical changes are not proportional to the pressure applied. It was remarked that the electrical effect, like the optical one, is dependent rather on the velocity than on its amplitude, and consequently we may conclude that the effect is due to the induced flow.

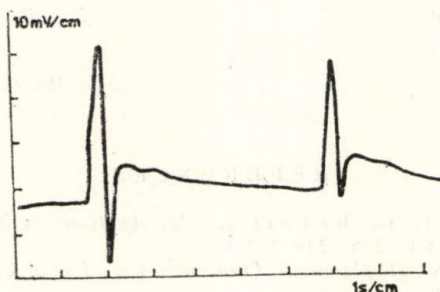


Fig. 3. — Changes of the polarized light transmittance of the LCD under mechanical stress.

The curves plotted (Fig. 4) show a long answer time and a fluctuating coming back process which may be explained by the amortization of the flow movement of the liquid crystal. The turn-off time seems to increase with the amplitude of the deformation.

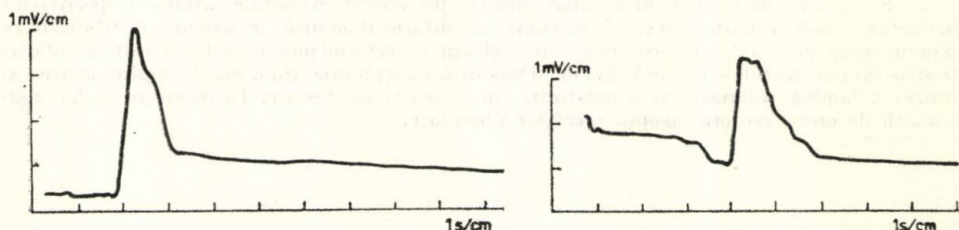


Fig. 4 — Changes of the electrical impedance (50 Hz) of the LCD under mechanical stress.

3. Discussions and Conclusions. 1. The optical, microscopic and macroscopic investigation, both in polarized and unpolarized light, as well as the electrical investigation proves the great sensibility of the nematic liquid crystals mechanically excited. The processes involve changes of the texture and of the electrical conductivity.

2. The sensitivity of the effects remarked for the two types of LCDs, having a liquid crystal layer of about 20 μm thick, is better than 5 μm (the measuring limit of the microcomparator used).

3. The macroscopical investigation as well as the curves plotted indicate that the changes noticed are due in the first place to the flow in the liquid crystal layer induced by the deformation of the structure of the device and on the other hand to the texture changes induced by the pressure. The last effect is visible only in polarized light.

4. The high sensitivity of the LCDs under the mechanical stress may be used in various configurations of detectors. The sharp answer, mainly in the optical processes, eliminates the problems due to parasitic effects induced by the ambient temperature changes. In the electrical measuring arrangements such parasitic effects may be important.

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EFECTELE PRESIUNII ASUPRA DISPOZITIVELOR CU CRISTALE LICHIDE

(Rezumat)

Se investighează în mod detaliat efectele presiunilor exercitate asupra dispozitivelor cu cristale lichide nematice, cu efect de cîmp sau difuzie dinamică, în scopul constituirii baze experimentale pentru dezvoltarea diverselor aplicații. Cercetările pun în evidență atât modificări de structură în stratul de cristal lichid cit și modificări ale proprietăților macroscopice electrice și optice, în lumină polarizată și nepolarizată, în dispozitivele testate. În încheiere se fac considerații de ordin teoretic asupra efectelor observate.

ABOUT THE MEAN NUMBER OF PHONONS

BY

MARIANA LATU

1. Introduction. In a solid the complex, collective oscillations of lattice ions is divided into a number of independent normal modes. These normal modes are quantized, and the associated quanta are called phonons. The phonons are bosons. Each state in the lattice vibration spectrum can therefore be occupied by any number of phonons. The number of phonons depends on the energy content of the lattice vibrations, i.e. on the temperature. At $T=0$ K, no phonons are excited and the lattice has its zero point energy only. The total number phonons is limited only of the stability of the crystal. If the thermal energy becomes too large the crystal structure changes or is broken up.

The mean number of phonons in state j , $\vec{k}$ is given by [1]

$$(1) \quad \langle n_j(\omega) \rangle = \frac{1}{e^{\frac{\hbar \omega_j(\vec{k})}{k_B T}} - 1},$$

where $\hbar \omega_j(\vec{k})$ is the energy of a phonon with the wave number $\vec{k}$ and the index of polarization j ; k_B is the Boltzmann constant. The function relating $\hbar \omega_j(\vec{k})$ and $\vec{k}$ is called the dispersion relation.

A crystal with s nonequivalent ions per primitive cell will have three branches (one mainly longitudinal, two mainly transversal) of its vibrational spectrum whose frequencies approach zero as $\vec{k} \rightarrow 0$. These branches are called acoustical modes. These will be in addition $3(s-1)$ branches each having a finite limiting frequency as $\vec{k} \rightarrow 0$; such branches are called optical modes.

For the acoustic phonons, in the limiting case of long wavelengths (small $\vec{k}$), $\omega_j(\vec{k})$ is proportional to k :

$$(2) \quad \omega_j(\vec{k}) = v_j k;$$

v_j is the velocity of elastic waves in crystal.

For isotropic media, the longitudinal or transversal velocity of sound is respectively [2]

$$(3) \quad v_l = \sqrt{\frac{C_{11}}{\rho}},$$

$$(4) \quad v_t = \sqrt{\frac{C_{44}}{\rho}},$$

where C_{11} and C_{44} are the elastic constants and ρ is the density of crystal.

For a cubic crystal [2]

$$(5) \quad v_{l[100]} = \sqrt{\frac{C_{11}}{\rho}},$$

$$(6) \quad v_{t[100]} = \sqrt{\frac{C_{44}}{\rho}},$$

$$(7) \quad v_{l[110]} = \sqrt{\frac{C_{11} + C_{12} + 2C_{44}}{2\rho}},$$

$$(8) \quad v_{t_1[110]} = \sqrt{\frac{C_{44}}{\rho}},$$

$$(9) \quad v_{t_2[110]} = \sqrt{\frac{C_{11} - C_{12}}{2\rho}}.$$

For an isotropic body the transverse velocity should be the same in all directions; so, from Eqs. (8) and (9) result

$$(10) \quad C_{11} - C_{12} = 2C_{44}.$$

In the case of an anisotropic crystal, the constant A defined by

$$(11) \quad A = \frac{2C_{44}}{C_{11} - C_{12}}$$

measures the departure from the isotropy.

For the optic phonons with $\vec{k} \rightarrow 0$, in the cubic crystal is valid the Lyddane - Sachs - Teller relation [3]

$$(12) \quad \frac{\omega_l^2}{\omega_t^2} = \frac{\epsilon_0}{\epsilon_\infty},$$

where ϵ_0 is the static dielectric constant and ϵ_∞ is the dielectric constant for very high frequency.

2. The Mecanical and Electric Properties of Crystal and the Mean Number of Phonons. We are interested to see how the ratio of the mean number of longitudinal phonons to the mean number of transversal phonons, acoustic or optic, depends on the mechanical and electric quantities. This relationships will give the information about the dominant type of phonons.

Using the relations (1), (2), (6) and (7), for cubic crystal, in the case of acoustic transversal phonons it results

$$(13) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{t_1[110]} \rangle} = \frac{e^{\frac{\hbar \sqrt{\frac{(C_{11} - C_{12})k}{2\rho}}}{k_B T}} - 1}{e^{\frac{\hbar \sqrt{\frac{C_{44}k}{\rho}}}{k_B T}} - 1}.$$

For $T \rightarrow 0$, 1 is neglected in respect to the exponential and therefore

$$(14) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle} = \frac{\frac{\hbar k}{k_B T} \sqrt{\frac{C_{44}}{\rho}} \left[\frac{1}{\sqrt{A}} - 1 \right]}{e}.$$

In the high-temperature region the expression (13) becomes

$$(15) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle} = \sqrt{\frac{C_{11} - C_{12}}{2C_{44}}} = \frac{1}{\sqrt{A}}.$$

For the longitudinal acoustic phonons, in the low-temperature region, using Eqs. (1), (2), (5) and (7) one obtains

$$(16) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle} = e^{\frac{\hbar k}{k_B T} \frac{1}{\sqrt{2\rho}} [\sqrt{2C_{11}} - \sqrt{C_{11} + C_{12} + 2C_{44}}]}$$

and in high-temperature region

$$(17) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle} = \sqrt{\frac{C_{11} + C_{12} + 2C_{44}}{2C_{11}}}.$$

For the direction $[100]$ (1), (2), (5) and (8), in low-temperature region we get

$$(18) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle} = e^{\frac{\hbar k}{k_B T} \frac{1}{\sqrt{\rho}} [\sqrt{C_{44}} - \sqrt{C_{11}}]}$$

and in high-temperature region

$$(19) \quad \frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle} = \sqrt{\frac{C_{44}}{C_{11}}}.$$

The Table 1 presents the ratios $\langle n_{l[100]} \rangle / \langle n_{l[110]} \rangle$, $\langle n_{l[100]} \rangle / \langle n_{l[110]} \rangle$ and $\langle n_{l[100]} \rangle / \langle n_{l[100]} \rangle$ for 12 alkali halides in the high-temperature region.

Table 1

Nr.	Crystal	C_{11} 10^9N/m^2	C_{44} 10^9N/m^2	C_{12} 10^9N/m^2	A	$\frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle}$	$\frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle}$	$\frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle}$
						$\frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle}$	$\frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle}$	$\frac{\langle n_{l[100]} \rangle}{\langle n_{l[110]} \rangle}$
1	LiF	112	63.5	46	1.92	0.72	0.72	1.13
2	NaF	97	28.1	24.2	0.77	1.14	0.54	0.96
3	NaCl	49.1	12.8	12.8	0.70	1.19	0.51	0.94
4	NaBr	40	9.96	10.6	0.68	1.21	0.50	0.94
5	KCl	40.5	6.27	6.9	0.37	1.64	0.39	0.86
6	KBr	34.5	5.10	5.5	0.37	1.69	0.38	0.85
7	KI	27.4	3.70	4.3	0.32	1.77	0.37	0.84
8	RbCl	36.4	4.70	6.3	0.31	1.79	0.36	0.85
10	RbBr	31.5	3.82	4.8	0.29	1.88	0.35	0.83
11	CsCl	36.6	8.07	9.0	0.59	1.31	0.47	0.92
12	CsBr	30.7	7.49	8.4	0.67	1.22	0.49	0.94

For optic phonons with $\vec{k} \rightarrow 0$, using the relations (1) and (2), we obtain for $T \rightarrow 0$ K

$$(20) \quad \frac{\langle n_l \rangle}{\langle n_t \rangle} = e^{\frac{\hbar \omega_l}{k_B T} \left(1 - \sqrt{\frac{\epsilon_g}{\epsilon_\infty}} \right)}$$

In the high-temperature region it results

$$(21) \quad \frac{\langle n_l \rangle}{\langle n_t \rangle} = \frac{\omega_t}{\omega_l} = \sqrt{\frac{\epsilon_\infty}{\epsilon_0}}$$

The Table 2 presents the ratio $\langle n_l \rangle / \langle n_t \rangle$ for optic phonons in the high-temperature region for the same twelve alkali halides.

Table 2

Nr.	Crystal	ϵ_∞	ϵ_0	$\frac{\langle n_l \rangle}{\langle n_t \rangle}$
1	LiF	1.9	8.9	0.46
2	NaF	1.7	5.1	0.58
3	NaCl	2.25	5.9	0.62
4	NaBr	2.6	6.4	0.64
5	KCl	2.1	4.85	0.66
6	KBr	2.3	4.9	0.68
7	KI	2.7	5.1	0.73
8	RbCl	2.2	4.9	0.67
9	RbBr	2.3	4.9	0.68
10	RbI	2.6	5.5	0.69
11	CsCl	2.6	7.2	0.60
12	CsBr	2.8	6.5	0.66

In the case of optic phonons with finite wave vector the Lyddane—Sachs—Teller relation for the direction [100] in the cubic crystals is [4]

$$(22) \quad \frac{\epsilon_l^s}{\epsilon_l^\infty} = \frac{\omega_{la}^2(\vec{k}) \omega_{t0}^2(\vec{k})}{\lambda_3^2(\vec{k}) \lambda_6^2(\vec{k})}$$

and for the direction [110]

$$(23) \quad \frac{\epsilon_t^s}{\epsilon_t^\infty} = \frac{\Omega_2^2(\vec{k}) \Omega_5^2(\vec{k})}{\omega_{ta}^2(\vec{k}) \omega_{t0}^2(\vec{k})}$$

In (22), (23), $\lambda_3^2(\vec{k})$ and $\lambda_6^2(\vec{k})$ are the poles of $\epsilon_l(\vec{k})$ while $\Omega_2^2(\vec{k})$ and $\Omega_5^2(\vec{k})$ are the zeroes of $\epsilon_t(\vec{k})$. At $k=0$, $\epsilon_l(\omega) = \epsilon_t(\omega)$ the zeroes of $\epsilon_l(\omega)$ (and therefore the zeroes of $\epsilon_t(\omega)$) correspond to the frequencies of $k=0$ LO phonons and the poles of $\epsilon_l(\omega)$ (and therefore the poles of $\epsilon_t(\omega)$) correspond to the frequencies of $k=0$ TO phonons. Thus at $\vec{k}=0$ we recover the usual Lyddane—Sachs—Teller relation (12).

Using Eqs. (1), (22) and (23) for the high-temperature region we get

$$(24) \quad n_{ia}^2 n_{io}^2 = \frac{k^4 T^4}{\hbar^2 (\epsilon_i^s / \epsilon_i^\infty) \lambda_3^2(\vec{k}) \lambda_6^2(\vec{k})},$$

$$(25) \quad n_{ia}^2 n_{io}^2 = \frac{k^4 T^4 \Omega_2^2(\vec{k}) \Omega_5^2(\vec{k})}{\hbar^4 (\epsilon_i^s / \epsilon_i^\infty)}.$$

3. Conclusions. 1. The relations (15), (16), (19) indicate that the ratio of the mean number of phonons of the miscellaneous types do not depend directly on the temperature. The ratio deped on the temperature by agency of the temperature dependence of second-order elastic constants. For the cubic crystals the ratio of the mean number of acoustic transversal phonons in the direction [100] to those in the direction [110] depends only by the constant A and thus this ratio is a measure of the degree of mechanical anisotropy.

2. From Table 1 results that for the 11 alkali halides, in the high-temperature region, $\langle n_{t[100]} \rangle > \langle n_{t[110]} \rangle$. For all alkali halides the mean number of acoustic longitudinal phonons ($\vec{k} \rightarrow 0$) in the direction [100] is more smaller than the mean number of acoustic transversal phonons ($\vec{k} \rightarrow 0$) in the same direction. The same fact results from the Table 2 for optic phonons in the limiting case of long wavelengths.

3. With the exception of Li F, from Tables 1 and 2 results, in the high-temperature region, a more large percentage of acoustic transversal phonons (in relation to the total mean number of acoustic phonons) in relation to that of optic transversal phonons (in relation to the total mean number of optic phonons).

The observable values were taken from [5] for C_{11} , C_{12} , C_{44} and from [1] for ϵ_0 and ϵ_∞ .

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ASUPRA NUMĂRULUI MEDIU DE FONONI

(Rezumat)

Se obțin relații de legătură între rapoartele numărului mediu de fononi acustici sau optici, după diverse direcții și mărimi ce caracterizează proprietățile mecanice și electrice ale cristalelor (constanta elastică de ordin doi și constanta dielectrică). Toate relațiile sînt valabile numai pentru cristale cubice; ele permit compararea cantitativă a diverselor tipuri de fononi.



$$\frac{A_{12}}{A_{11}} = \frac{A_{12}^0}{A_{11}^0} \frac{C_{12}(T)}{C_{11}(T)} \frac{C_{11}^0}{C_{12}^0}$$

(24)

$$\frac{A_{13}}{A_{11}} = \frac{A_{13}^0}{A_{11}^0} \frac{C_{13}(T)}{C_{11}(T)} \frac{C_{11}^0}{C_{13}^0}$$

(25)

3. Conclusions. The relations (15), (16), (19) indicate that the ratio of the mean number of phonons of the miscellaneous types do not depend directly on the temperature. The ratio depends on the temperature by agency of the temperature dependence of second-order elastic constants. For the cubic crystals the ratio of the mean number of acoustic transversal phonons in the direction [100] to those in the direction [110] depends only by the constant k and thus this ratio is a measure of the degree of mechanical anisotropy.

2. From Table 1 results that for the 11 alkali halides in the high-temperature region, $\langle n_{\text{trans}} \rangle > \langle n_{\text{long}} \rangle$. For all alkali halides the mean number of acoustic longitudinal phonons ($\lambda=0$) in the direction [100] is more smaller than the mean number of acoustic transversal phonons ($\lambda=0$) in the same direction. The same fact results from the Table 2 for optic phonons in the limiting case of long wavelengths.

3. With the exception of LiF, from Table 1 and 2 results in the high-temperature region, a more large percentage of acoustic transversal phonons in relation to the total mean number of acoustic phonons in relation to that of optic transversal phonons in relation to the total mean number of optic phonons.

The observed values were taken from [5] for C_{11} , C_{12} and from [1] for c_{12} and c_{44} .

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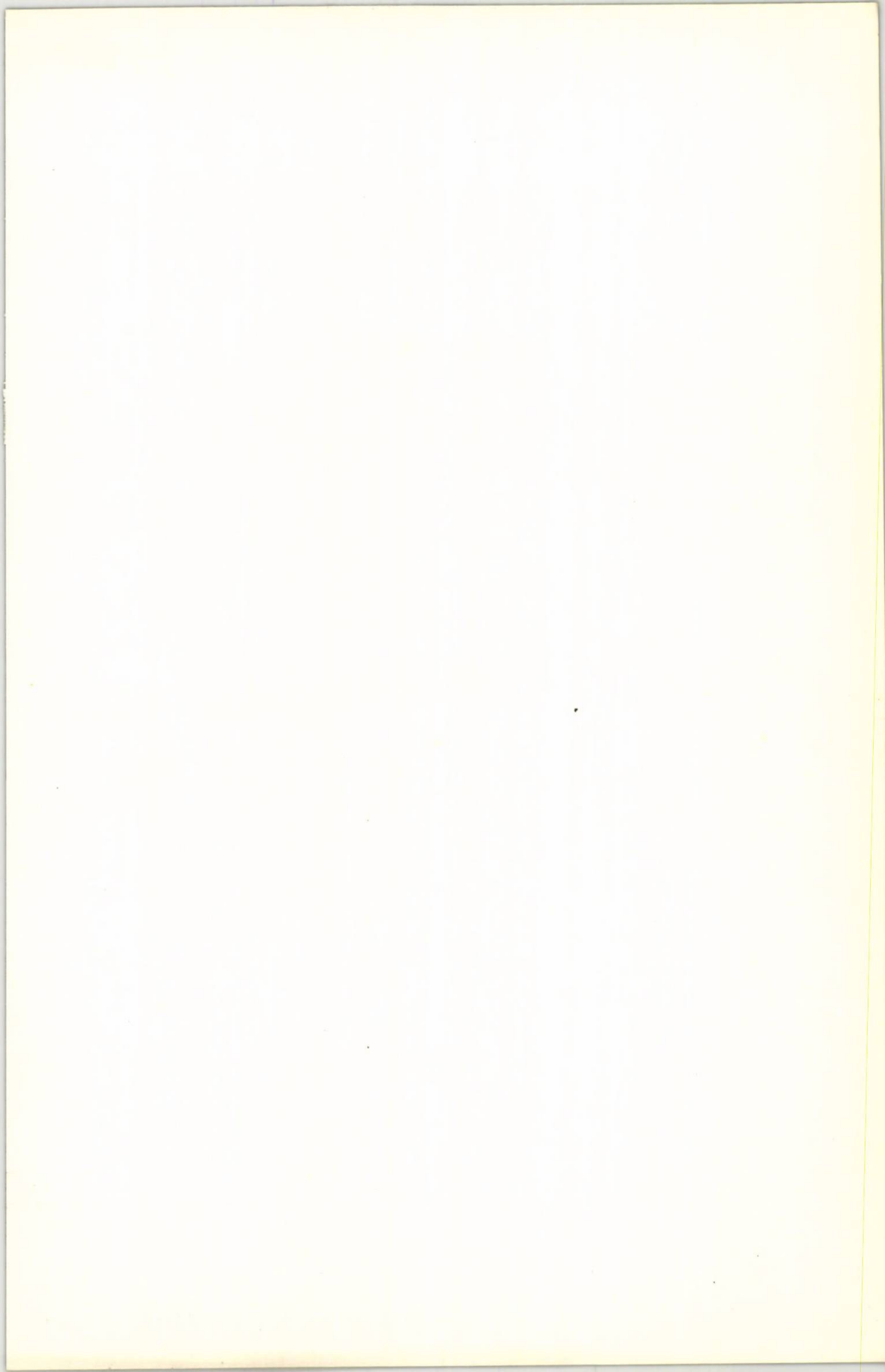
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REZUMATUL LUCRĂRII ÎN LIMBA ROMÂNĂ

(Rezumat)

Se calculează raportul dintre numărul mediu de fononi acustici transversali și longitudinali în funcție de temperatură pentru 11 haluri alcaline în regiunea de înaltă temperatură. Pentru toate halurile alcaline rezultă că numărul mediu de fononi acustici transversali este mai mic decât numărul mediu de fononi acustici longitudinali în direcția [100].



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